

SNAKE GRAPH CALCULUS AND CLUSTER ALGEBRAS FROM SURFACES

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ABSTRACT. Snake graphs appear naturally in the theory of cluster algebras. For cluster algebras from surfaces, each cluster variable is given by a formula which is parametrized by the perfect matchings of a snake graph. In this paper, we identify each cluster variable with its snake graph, and interpret relations among the cluster variables in terms of these graphs. In particular, we give a new proof of skein relations of two cluster variables.

1. INTRODUCTION

Cluster algebras were introduced in [FZ1], and further developed in [FZ2, BFZ, FZ4], motivated by combinatorial aspects of canonical bases in Lie theory [L1, L2]. A cluster algebra is a subalgebra of a field of rational functions in several variables, and is given by constructing a distinguished set of generators, the *cluster variables*. The cluster variables are constructed recursively and their computation is rather complicated in general. By construction cluster variables are rational functions, but it was shown in [FZ1] that they are actually Laurent polynomials with integer coefficients. Moreover, these coefficients are conjectured to be positive; this is the positivity conjecture.

An important class of cluster algebras is given by the cluster algebras from surfaces [GSV, FG1, FG2, FST, FT]. From a classification point of view, this class is very important since it has been shown in [FeShTu] that almost all (skew-symmetric) mutation finite cluster algebras are from surfaces. For generalizations to the skew-symmetrizable case, see [FeShTu2, FeShTu3]. If \mathcal{A} is a cluster algebra from a surface, then there exists a marked surface with boundary, such that the cluster variables of \mathcal{A} are in bijection with certain curves, called *arcs*, in the surface and the relations between the cluster variables are given by the crossing patterns of the arcs in the surface.

In a collaboration with Musiker and Williams [MSW], building on earlier work [S2, ST, S3, MS], the second author used this geometric interpretation to obtain a direct combinatorial formula for the cluster variables in cluster algebras from surfaces. This formula is manifestly positive and thus proves the positivity conjecture. In [MSW2], the formula was the key ingredient in the construction of two bases for the cluster algebra in the case where the surface has no punctures.

The formula is parametrized by perfect matchings of certain graphs, the *snake graphs*, which are the subject of the present paper. Snake graphs had appeared earlier in [Pr] in the special case of triangulated polygons. To compute the cluster variable associated to an arc γ , one constructs the snake graph \mathcal{G}_γ as a union of square shaped graphs, called *tiles*, which correspond to quadrilaterals in a fixed triangulation of the surface: one tile for each quadrilateral traversed by γ . These tiles are glued together according to the geometry of the surface. Thus to every cluster variable corresponds an arc, and to every arc corresponds a snake graph. A natural question is then how much of the relations among the cluster variables or, equivalently, how much of the geometry of the arcs, can be recovered from the snake graphs alone?

For example, the product of two cluster variables can be thought of as the union of the two corresponding snake graphs. It is known [MW, MSW2] that if the two arcs cross, then one can rewrite the product of the cluster variables as a linear combination of elements corresponding to arcs without crossings. The process of resolving the crossings is given on the arcs by a *smoothing*

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operation and the relation in the cluster algebra is called a *skein relation*. On the level of snake graphs, one needs to know the following.

- (1) When do the two arcs corresponding to two snake graphs cross?
- (2) What are the snake graphs corresponding to the skein relations?

In this paper, we introduce the notion of an *abstract snake graph*, which is not necessarily related to an arc in a surface. Then we define what it means for two abstract snake graphs to cross. Given two crossing snake graphs, we construct the resolution of the crossing as two pairs of snake graphs from the original pair of crossing snake graphs. We then prove that there is a bijection φ between the set of perfect matchings of the two crossing snake graphs and the set of perfect matchings of the resolution, Theorem 3.1.

We then apply our constructions to snake graphs arising from unpunctured surfaces and prove that

- (1) two snake graphs cross if and only if the corresponding arcs cross in the surface, Theorem 5.3;
- (2) the resolution of the crossing of the snake graphs coincides with the snake graphs of the curves obtained from the two crossing arcs by smoothing, Theorems 5.5 and 5.8;
- (3) the bijection φ above is weight preserving, where the weights of the edges are the initial cluster variables associated to the arcs of the initial triangulation of the surface. Consequently, the Laurent polynomials associated to a pair of crossing snake graphs and to its resolution are identical, Theorems 6.1 and 6.3.

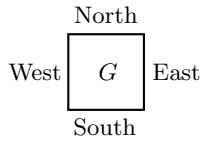
As an application, we obtain a combinatorial formula in terms of snake graphs for the product of cluster variables, and a new proof of the skein relations for cluster variables, Corollary 6.6.

The paper is organized as follows. We introduce the notions of abstract snake graphs, as well as their crossings and resolutions in section 2. In section 3, we give the definition of the bijection φ . After recalling the definitions and results on cluster algebras from surfaces in section 4, we prove our results (1)-(3) in sections 5 and 6. Section 7 contains the proof that the map φ is a bijection.

2. ABSTRACT SNAKE GRAPHS

Fix an orthonormal basis of the plane.

A *tile* G is a square of fixed side-length in the plane whose sides are parallel or orthogonal to the fixed basis.



We consider a tile G as a graph with four vertices and four edges in the obvious way. A *snake graph* \mathcal{G} is a connected graph consisting of a finite sequence of tiles G_1, G_2, \dots, G_d with $d \geq 1$, such that for each $i = 1, \dots, d-1$

- (i) G_i and G_{i+1} share exactly one edge e_i and this edge is either the north edge of G_i and the south edge of G_{i+1} or the east edge of G_i and the west edge of G_{i+1} .
- (ii) G_i and G_j have no edge in common whenever $|i - j| \geq 2$.
- (ii) G_i and G_j are disjoint whenever $|i - j| \geq 3$.

An example is given in Figure 1.

We sometimes use the notation $\mathcal{G} = (G_1, G_2, \dots, G_d)$ for the snake graph and $\mathcal{G}[i, i+t] = (G_i, G_{i+1}, \dots, G_{i+t})$ for the subgraph of \mathcal{G} consisting of the tiles $G_i, G_{i+1}, \dots, G_{i+t}$.

The $d-1$ edges e_1, e_2, \dots, e_{d-1} which are contained in two tiles are called *interior edges* of \mathcal{G} and the other edges are called *boundary edges*.

By convention we call a set containing a single edge an *empty snake graph*.

A snake graph \mathcal{G} is called *straight* if all its tiles lie in one column or one row, and a snake graph is called *zigzag* if no three consecutive tiles are straight.

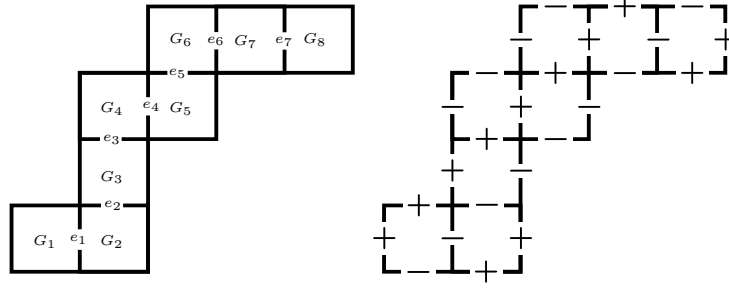


FIGURE 1. A snake graph with 8 tiles and 7 interior edges (left); a sign function on the same snake graph (right)

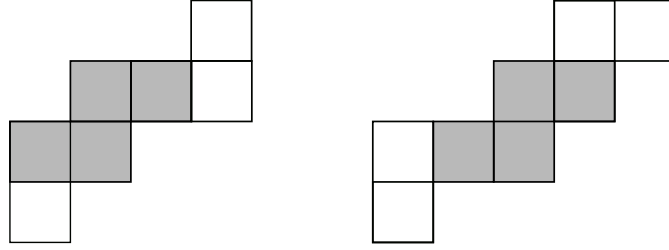


FIGURE 2. Two snake graphs with overlap (shaded)

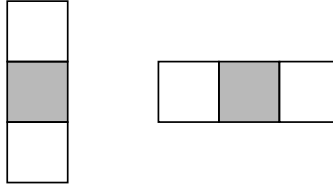


FIGURE 3. Two snake graphs with overlap consisting of a single tile (shaded)

2.1. Sign function. A *sign function* f on a snake graph \mathcal{G} is a map f from the set of edges of \mathcal{G} to $\{+, -\}$ such that on every tile in \mathcal{G} the north and the west edge have the same sign, the south and the east edge have the same sign and the sign on the north edge is opposite to the sign on the south edge. See Figure 1 for an example.

Note that on every nonempty snake graph there are exactly two sign functions.

2.2. Overlaps. Let $\mathcal{G}_1 = (G_1, G_2, \dots, G_d)$ and $\mathcal{G}_2 = (G'_1, G'_2, \dots, G'_{d'})$ be two snake graphs. We say that \mathcal{G}_1 and \mathcal{G}_2 have an *overlap* \mathcal{G} if \mathcal{G} is a snake graph and there exist two embeddings $i_1 : \mathcal{G} \rightarrow \mathcal{G}_1$, $i_2 : \mathcal{G} \rightarrow \mathcal{G}_2$ which are maximal in the following sense.

- (i) If \mathcal{G} has at least two tiles and if there exists a snake graph \mathcal{G}' with two embeddings $i'_1 : \mathcal{G}' \rightarrow \mathcal{G}_1$, $i'_2 : \mathcal{G}' \rightarrow \mathcal{G}_2$ such that $i_1(\mathcal{G}) \subseteq i'_1(\mathcal{G}')$ and $i_2(\mathcal{G}) \subseteq i'_2(\mathcal{G}')$ then $i_1(\mathcal{G}) = i'_1(\mathcal{G}')$ and $i_2(\mathcal{G}) = i'_2(\mathcal{G}')$.
- (ii) If \mathcal{G} consists of a single tile then using the notation $G_k = i_1(\mathcal{G})$ and $G'_{k'} = i_2(\mathcal{G})$, we have
 - (a) $k \in \{1, d\}$ or $k' \in \{1, d'\}$ or
 - (b) $1 < k < d$, $1 < k' < d'$ and the subgraphs (G_{k-1}, G_k, G_{k+1}) and $(G'_{k'-1}, G'_{k'}, G'_{k'+1})$ are either both straight or both zigzag subgraphs.

An example of type (i) is shown in Figure 2 and an example of type (ii)(b) in Figure 3.

Note that two snake graphs may have several overlaps with respect to different snake graphs \mathcal{G} .

2.3. Crossing. Let $\mathcal{G}_1 = (G_1, G_2, \dots, G_d)$, $\mathcal{G}'_1 = (G'_1, G'_2, \dots, G'_{d'})$ be two snake graphs with overlap \mathcal{G} and embeddings $i_1(\mathcal{G}) = (G_s, G_{s+1}, \dots, G_t)$ and $i_2(\mathcal{G}) = (G'_{s'}, G'_{s'+1}, \dots, G'_{t'})$ and suppose without loss of generality that $s \leq t$ and $s' \leq t'$. Let e_1, \dots, e_{d-1} (respectively $e'_1, \dots, e'_{d'-1}$) be the interior edges of \mathcal{G}_1 (respectively \mathcal{G}'_1 .) Let f be a sign function on \mathcal{G} . Then f induces a sign function f_1 on \mathcal{G}_1 and f_2 on \mathcal{G}'_1 . Moreover, since the overlap \mathcal{G} is maximal, we have

$$\begin{aligned} f_1(e_{s-1}) &= -f_2(e'_{s'-1}) & \text{if } s > 1, s' > 1 \\ f_1(e_t) &= -f_2(e'_{t'}) & \text{if } t < d, t' < d'. \end{aligned}$$

Definition 2.1. We say that \mathcal{G}_1 and \mathcal{G}_2 cross in \mathcal{G} if one of the following conditions hold.

(i)

$$\begin{aligned} f_1(e_{s-1}) &= -f_1(e_t) & \text{if } s > 1, t < d \\ \text{or} \\ f_2(e'_{s'-1}) &= -f_2(e'_{t'}) & \text{if } s' > 1, t' < d' \end{aligned}$$

(ii)

$$\begin{aligned} f_1(e_t) &= f_2(e'_{s'-1}) & \text{if } s = 1, t < d, s' > 1, t' = d' \\ \text{or} \\ f_1(e_{s-1}) &= f_2(e'_{t'}) & \text{if } s > 1, t = d, s' = 1, t' < d' \end{aligned}$$

Remark 2.2. 1. The definition does not depend on the choice of the sign function f .
 2. \mathcal{G}_1 and \mathcal{G}_2 may still cross if $s = 1$ and $t = d$ because they may satisfy condition 2.1(i).
 3. The terminology ‘cross’ comes from snake graphs that are associated to arcs in a surface. We shall show in Theorem 5.3 that two such arcs cross if and only if the corresponding snake graphs cross in an overlap.

2.4. Resolution of crossing. Given two snake graphs that cross, we construct two pairs of new snake graphs, which we call the resolution of the crossing. In section 3, we show that there is a bijection between the set of perfect matchings of the two crossing snake graphs and the set of perfect matchings of the resolution. In sections 5 and 6, we show that this construction is related to multiplication formulas given by Skein relations in cluster algebras.

Let $\mathcal{G}_1, \mathcal{G}_2$ be two snake graphs crossing in an overlap $\mathcal{G} = \mathcal{G}_1[s, t] = \mathcal{G}_2[s', t']$. Recall that $\mathcal{G}_k[i, j]$ is the subgraph of \mathcal{G}_k given by the tiles with indices $i, i+1, \dots, j$. Let $\bar{\mathcal{G}}_k[j, i]$ be the snake graph obtained by reflecting $\mathcal{G}_k[i, j]$ such that the order of the tiles is reversed.

We define four connected subgraphs as follows, see Figure 4 for examples.

$\mathcal{G}_3 = \mathcal{G}_1[1, t] \cup \mathcal{G}_2[t' + 1, d']$ where the adjacency of the two subgraphs is induced by \mathcal{G}_2 .

$\mathcal{G}_4 = \mathcal{G}_2[1, t'] \cup \mathcal{G}_1[t + 1, d]$ where the adjacency of the two subgraphs is induced by \mathcal{G}_1 .

$$\mathcal{G}_5 = \begin{cases} \mathcal{G}_1[1, s-1] \cup \bar{\mathcal{G}}_2[s'-1, 1] & \text{if } s > 1, s' > 1 \text{ where the two subgraphs are glued} \\ & \text{along the north of } G_{s-1} \text{ and the east of } G'_{s'-1} \text{ if } G_s \\ & \text{is east of } G_{s-1} \text{ in } \mathcal{G}_1; \text{ and along the east of } G_{s-1} \text{ and} \\ & \text{the north of } G'_{s'-1} \text{ if } G_s \text{ is north of } G_{s-1} \text{ in } \mathcal{G}_1; \\ \mathcal{G}_1[1, k] & \text{if } s' = 1 \text{ where } k < s-1 \text{ is the largest integer such} \\ & \text{that } f_1(e_k) = f_1(e_{s-1}) \text{ if such a } k \text{ exists;} \\ \{e_0\} & \text{if } s' = 1 \text{ and no such } k \text{ exists, where } e_0 \text{ is the} \\ & \text{unique edge of } G_1 \text{ which is south or west and sat-} \\ & \text{sifies } f_1(e_0) = f_1(e_{s-1}); \\ \bar{\mathcal{G}}_2[k', 1] & \text{if } s = 1 \text{ where } k' < s'-1 \text{ is the largest integer such} \\ & \text{that } f_2(e_{k'}) = f_2(e_{s'-1}) \text{ if such a } k' \text{ exists;} \\ \{e'_0\} & \text{if } s = 1 \text{ and no such } k' \text{ exists, where } e'_0 \text{ is the} \\ & \text{unique edge of } G'_1 \text{ which is south or west and sat-} \\ & \text{sifies } f_2(e'_0) = f_2(e'_{s'-1}); \end{cases}$$

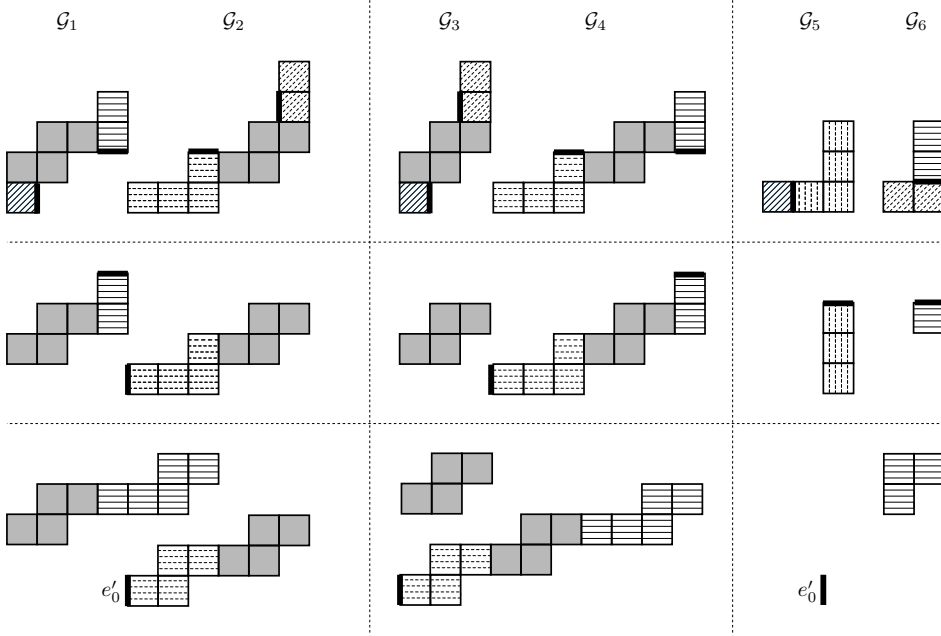


FIGURE 4. Examples of resolutions: $s > 1, s' > 1, t < d, t' < d'$ in the first row; $s = 1, t' = d'$ in the second and third row;

$$\mathcal{G}_6 = \begin{cases} \overline{\mathcal{G}}_2[d', t' + 1] \cup \mathcal{G}_1[t + 1, d] & \text{if } t < d, t' < d' \text{ where the two subgraphs are glued} \\ & \text{along the west of } G_{t+1} \text{ and the south of } G'_{t'+1} \text{ if } G_{t+1} \\ & \text{is north of } G_t \text{ in } \mathcal{G}_1; \text{ and along the south of } G_{t+1} \text{ and} \\ & \text{the west of } G'_{t'+1} \text{ if } G_{t+1} \text{ is east of } G_t \text{ in } \mathcal{G}_1; \\ \overline{\mathcal{G}}_2[d', k'] & \text{if } t = d \text{ where } k' > t' + 1 \text{ is the least integer such that} \\ & f_2(e'_{t'}) = f_2(e'_{k'-1}) \text{ if such a } k' \text{ exists;} \\ \{e'_{d'}\} & \text{if } t = d \text{ and no such } k' \text{ exists, where } e'_{d'} \text{ is the} \\ & \text{unique edge of } G'_{d'} \text{ which is north or east and sat-} \\ & \text{satisfies } f_2(e'_{d'}) = f_2(e'_{t'}); \\ \mathcal{G}_1[k, d] & \text{if } t' = d' \text{ where } k > t + 1 \text{ is the least integer such that} \\ & f_1(e_t) = f_1(e_{k-1}) \text{ if such a } k \text{ exists;} \\ \{e_d\} & \text{if } t' = d' \text{ and no such } k \text{ exists, where } e_d \text{ is the} \\ & \text{unique edge of } G_d \text{ which is north or east and satisfies} \\ & f_1(e_d) = f_1(e_t); \end{cases}$$

Definition 2.3. In the above situation, we say that the pair $(\mathcal{G}_3 \sqcup \mathcal{G}_4, \mathcal{G}_5 \sqcup \mathcal{G}_6)$ is the resolution of the crossing of \mathcal{G}_1 and \mathcal{G}_2 at the overlap \mathcal{G} and we denote it by $\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$.

If $\mathcal{G}_1, \mathcal{G}_2$ have no crossing in \mathcal{G} we let $\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2) = \mathcal{G}_1 \sqcup \mathcal{G}_2$.

Remark 2.4. The pair $(\mathcal{G}_3, \mathcal{G}_4)$ still has an overlap in \mathcal{G} but without crossing. The pair $(\mathcal{G}_5, \mathcal{G}_6)$ can be thought of as a reduced symmetric difference of \mathcal{G}_1 and \mathcal{G}_2 with respect to the overlap \mathcal{G} .

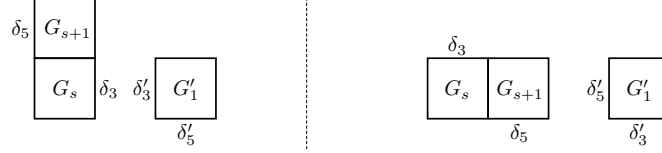
2.5. Grafting. In this subsection, we define another operation which to two snake graphs associates two pairs of snake graphs. Here however, we do not suppose that the original two snake graphs have an overlap. In section 3, we show that there is a bijection between the set of perfect matchings of the two original snake graphs and the set of perfect matchings of the two new pairs. In sections 5 and 6, we show that this construction is related to multiplication formulas in cluster algebras.

Let $\mathcal{G}_1 = (G_1, G_2, \dots, G_d)$, $\mathcal{G}_2 = (G'_1, G'_2, \dots, G'_{d'})$ be two snake graphs and let f_1 be a sign function on \mathcal{G}_1 .

Case 1. Let s be such that $1 < s < d$.

If G_{s+1} is north of G_s in \mathcal{G}_1 then let δ_3 denote the east edge of G_s , δ_5 the west edge of G_{s+1} and δ'_3 the west edge of G'_1 , δ'_5 the south edge of G'_1 .

If G_{s+1} is east of G_s in \mathcal{G}_1 , then let δ_3 denote the north edge of G_s , δ_5 the south edge of G_{s+1} and δ'_3 the south edge of G'_1 , δ'_5 the west edge of G'_1 . Thus we have one of the following two situations.



Define four snake graphs as follows; see Figure 5 for examples.

$\mathcal{G}_3 = \mathcal{G}_1[1, s] \cup \mathcal{G}_2$ where the two subgraphs are glued along the edges δ_3 and δ'_3 .

$$\mathcal{G}_4 = \begin{cases} \mathcal{G}_1[k_4, d] & \text{where } k_4 > s+1 \text{ is the least integer such that } f_1(e_s) = -f_1(e_{k_4-1}) \text{ if such a } k_4 \text{ exists;} \\ \{e_d\} & \text{otherwise where } e_d \text{ is the unique edge of } G_d \text{ which is north or east and such that } f_1(e_s) = -f_1(e_d) \end{cases}$$

$$\mathcal{G}_5 = \begin{cases} \mathcal{G}_1[1, k_5] & \text{where } k_5 < s \text{ is the largest integer such that } f_1(e_{k_5}) = -f_1(e_s) \text{ if such a } k_5 \text{ exists;} \\ \{e_0\} & \text{otherwise where } e_0 \text{ is the unique edge of } G_1 \text{ which is south or west and such that } f_1(e_0) = -f_1(e_s) \end{cases}$$

$\mathcal{G}_6 = \overline{\mathcal{G}}_2[d', 1] \cup \mathcal{G}_1[s+1, d]$ where the two subgraphs are glued along the edges δ_5 and δ'_5 .

Case 2. Now let $s = d$. Choose a pair of edges (δ_3, δ'_3) such that either δ_3 is the north edge in G_s and δ'_3 is the south edge in G'_1 or δ_3 is the east edge in G_s and δ'_3 is the west edge in G'_1 . Let f_2 be a sign function on \mathcal{G}_2 such that $f_2(\delta'_3) = f_1(\delta_3)$. Then define four snake graphs as follows.

$\mathcal{G}_3 = \mathcal{G}_1[1, s] \cup \mathcal{G}_2$ where the two subgraphs are glued along the edges δ_3 and δ'_3 .

$\mathcal{G}_4 = \{\delta_3\}$

$$\mathcal{G}_5 = \begin{cases} \mathcal{G}_1[1, k_5] & \text{where } k_5 < s \text{ is the largest integer such that } f_1(e_{k_5}) = f_1(\delta_3), \text{ if such a } k_5 \text{ exists;} \\ \{e_0\} & \text{otherwise, where } e_0 \text{ is the unique edge of } G_1 \text{ which is south or west and such that } f_1(e_0) = f_1(\delta_3) \end{cases}$$

$$\mathcal{G}_6 = \begin{cases} \overline{\mathcal{G}}_2[d', k_6] & \text{where } k_6 > 1 \text{ is the least integer such that } f_2(e'_{k_6-1}) = f_1(\delta_3), \text{ if such a } k_6 \text{ exists;} \\ \{e_0\} & \text{otherwise, where } e'_{d'} \text{ is the unique edge of } G'_{d'} \text{ which is south or west and such that } f_1(e'_{d'}) = f_1(\delta_3). \end{cases}$$

Definition 2.5. In the above situation, we say that the pair $(\mathcal{G}_3 \sqcup \mathcal{G}_4, \mathcal{G}_5 \sqcup \mathcal{G}_6)$ is called the resolution of the grafting of \mathcal{G}_2 on \mathcal{G}_1 in \mathcal{G}_s and we denote it by $\text{Graft}_{s, \delta_3}(\mathcal{G}_1, \mathcal{G}_2)$.

3. PERFECT MATCHINGS

Recall that a *perfect matching* P of a graph G is a subset of the set of edges of G such that each vertex of G is incident to exactly one edge in P . Let $\text{Match}(G)$ denote the set of all perfect matchings of the graph G .

The main result of this section is the following.

Theorem 3.1. Let $\mathcal{G}_1, \mathcal{G}_2$ be two snake graphs. Then there are bijections

$$\begin{aligned} (1) \quad & \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2) \xrightarrow{\varphi=(\varphi_{34}, \varphi_{56})} \text{Match}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1 \sqcup \mathcal{G}_2)) \\ (2) \quad & \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2) \xrightarrow{\varphi=(\varphi_{34}, \varphi_{56})} \text{Match}(\text{Graft}_{s, e_3}(\mathcal{G}_1 \sqcup \mathcal{G}_2)) \end{aligned}$$

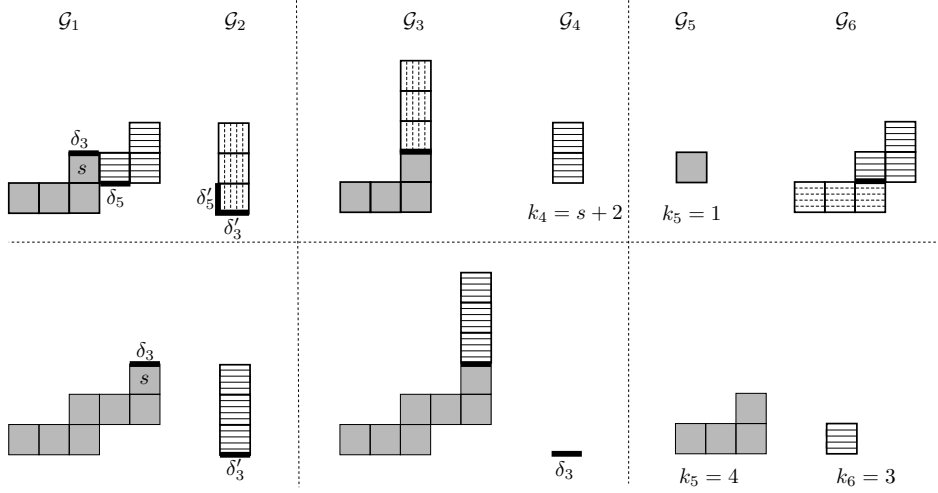


FIGURE 5. Examples of resolutions of graftings: $1 < s < d$ in the first row; $s = d'$ in the second row.

Proof. We shall explicitly construct the bijection φ . The proof that φ is a bijection is given in section 7.

The idea for the bijection φ is simple: Look for the first place in the overlap where one can ‘switch’ the matchings of \mathcal{G}_1 and \mathcal{G}_2 in order to get a matching of $\mathcal{G}_3 \sqcup \mathcal{G}_4$. If the given matching does not have such a switching place at all, then its image under φ is a matching of $\mathcal{G}_5 \sqcup \mathcal{G}_6$.

If P is a matching of the snake graph \mathcal{G} , we denote by $P[j_1, j_2]$ its restriction to the subgraph $\mathcal{G}[j_1, j_2]$, and we define

$$\begin{aligned} P(j_1, j_2) &= P[j_1, j_2] \setminus \{e_{j_1-1}\}, \\ P[j_1, j_2] &= P[j_1, j_2] \setminus \{e_{j_2}\}. \end{aligned}$$

We first define φ for statement (1). The statement is trivial if \mathcal{G}_1 and \mathcal{G}_2 do not have a crossing in \mathcal{G} . Suppose therefore \mathcal{G}_1 and \mathcal{G}_2 cross in \mathcal{G} and use the notation of the Definition 2.3 and Definition 2.5. Let $P_i \in \text{Match } \mathcal{G}_i$.

We define the map φ by the following procedure. Suppose first that $s, s' \neq 1, t \neq d, t' \neq d'$ and $s \neq t$.

- (i) If the pair of matchings $(P_1[s-1, s+1], P_2[s'-1, s'+1])$ on the pair of subgraphs $(\mathcal{G}_1[s-1, s+1], \mathcal{G}_2[s'-1, s'+1])$ is one of the eight configurations on the left in Figure 6 then let $\varphi(P_1, P_2)$ be

$$(P_1[1, s-1] \cup \mu_{s,1} \cup P_2(s'+1, d'], P_2[1, s'-1] \cup \mu_{s,2} \cup P_1(s+1, d])$$

- (ii) If (i) does not apply, let j be the least integer such that $1 < j < t-s-2$ and the local configuration of (P_1, P_2) on $(\mathcal{G}_1[s+j, s+j+1], \mathcal{G}_2[s'+j, s'+j+1])$ is one of the four configurations shown in Figure 8, if such j exists, and let $\varphi(P_1, P_2)$ be

$$\begin{aligned} &(P_1[1, s+j-1] \cup \rho_{j,1} \cup P_2(s'+j+2, d'], \\ &P_2[1, s'+j-1] \cup \rho_{j,2} \cup P_1(s+j+2, d]) \end{aligned}$$

- (iii) If (i) and (ii) do not apply and the local configuration of (P_1, P_2) on $(\mathcal{G}_1[t-1, t+1], \mathcal{G}_2[t'-1, t'+1])$ is one of the eight shown in Figure 6, relabeling $s = t, s-1 = t+1, s+1 = t-1, s' = t', s'-1 = t'+1, s'+1 = t'-1$, let $\varphi(P_1, P_2)$ be

$$(P_1[1, t-1] \cup \mu_{t,1} \cup P_2(t'+1, d'], P_2[1, t'-1] \cup \mu_{t,2} \cup P_1(t+1, d])$$

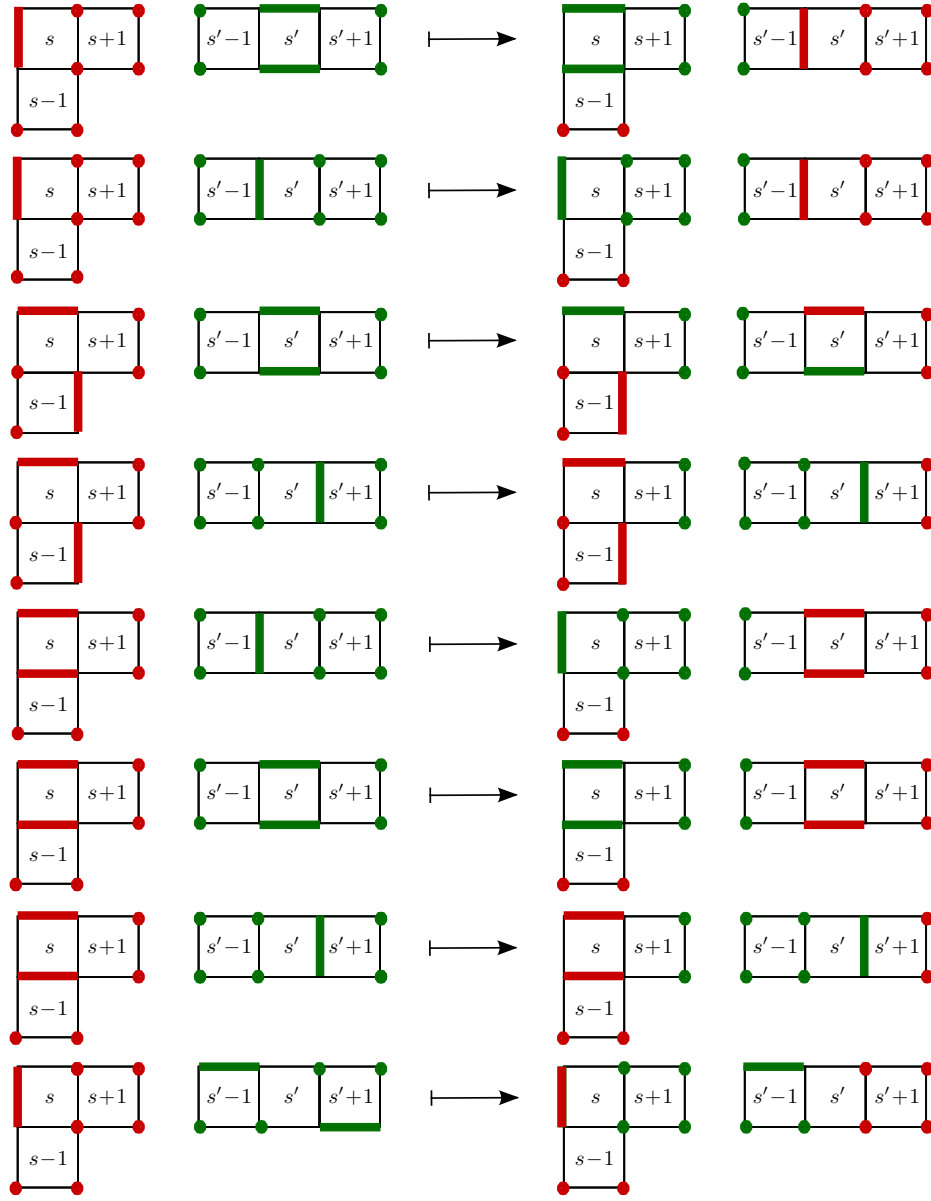


FIGURE 6. The operation μ applies in the 8 cases for $(P_1[s-1, s+1], P_2[s'-1, s'+1])$ shown on the left. Colored edges must belong to the matchings, colored vertices can be matched arbitrarily. The resulting pairs $(\mu_{s,1}, \mu_{s,2})$ are shown on the right. Colors indicate whether the edges belong to P_1 or P_2 .

- (iv) If (i)-(iii) do not apply then let a_5 (respectively a_6) be the glueing edge in the definition of \mathcal{G}_5 (respectively \mathcal{G}_6) in Definition 2.3. Then let $\varphi(P_1, P_2)$ be

$$(P_1[1, s-1] \sqcup P_2[1, s'-1] \setminus \{a_5\}, P_2[t'+1, d'] \sqcup P_1[t+2, d] \setminus \{a_6\})$$

where the notation $A \sqcup B \setminus \{a\}$ means

$$\begin{array}{ll} A \cup B \setminus \{a\} & \text{if } a \notin A \cap B \\ A \cup B & \text{if } a \in A \cap B \end{array}$$

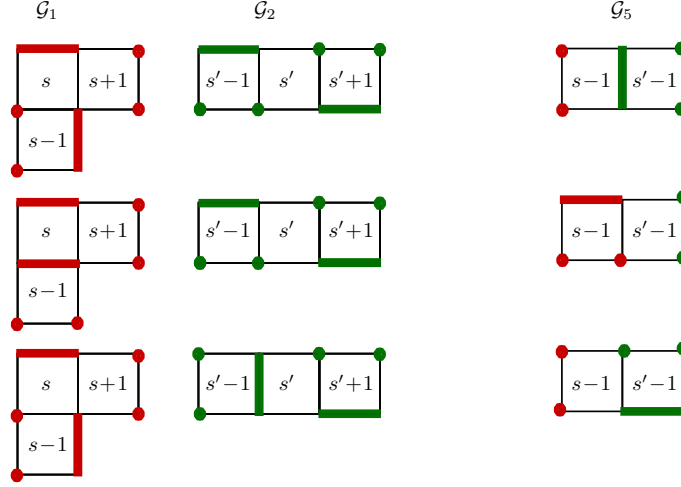
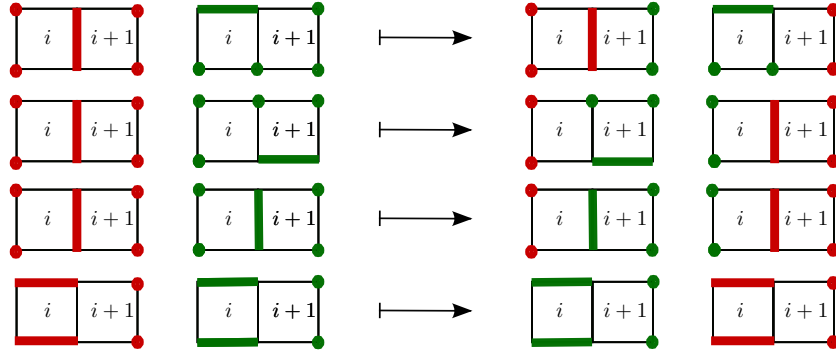

 FIGURE 7. The 3 cases in which the operation μ does not apply

 FIGURE 8. The operation ρ applies in the 4 cases shown on the left. The resulting pairs $(\rho_{i,1}, \rho_{i,2})$ are shown on the right.

 FIGURE 9. The 2 cases in which the operation ρ does not apply

Note that in cases (i)-(iii), $\varphi(P_1, P_2) \in \text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4)$ and in case (iv), $\varphi(P_1, P_2) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$.

If $s \neq t$ and, $s = 1$, $s' = 1$, $t = d$, or $t' = d'$ respectively, we define $\varphi(P_1, P_2)$ by following the steps (i)-(iv) ignoring tiles $G_{s-1}, G'_{s'-1}, G_{t+1}$ or $G'_{t'+1}$ respectively, and restricting to $\mathcal{G}_5 \cup \mathcal{G}_6$ if step (iv) has been applied.

Finally if $s = t$ then φ is defined by step (i) replacing the operation μ by the operation ν in Figure 10 and step (iv) only.

Now we define φ for the statement (2). Let $\sigma_s = (\sigma_{s,3}, \sigma_{s,5})$ be the map described in Figure 11. Then define

$$\varphi : \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2) \longrightarrow \text{Match}(\text{Graft}_{s,e_3}(\mathcal{G}_1, \mathcal{G}_2))$$

as follows.

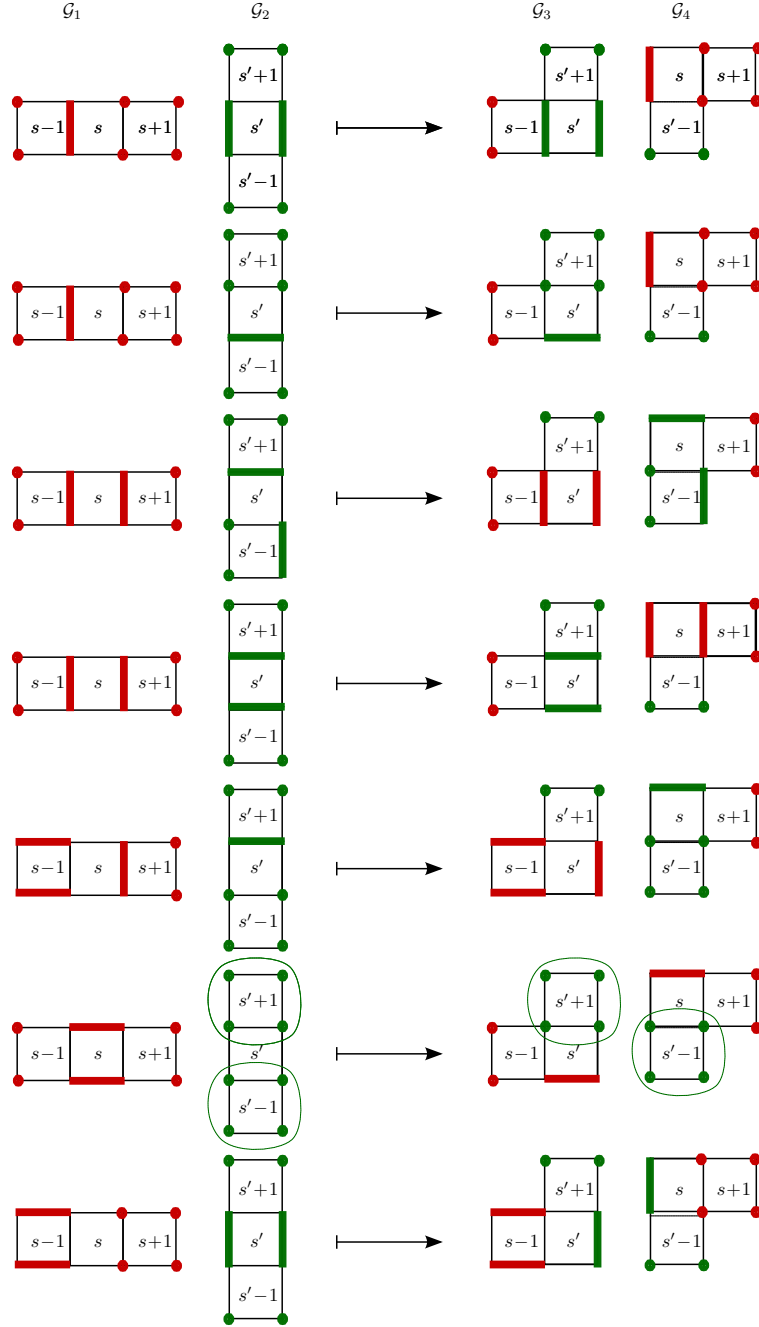
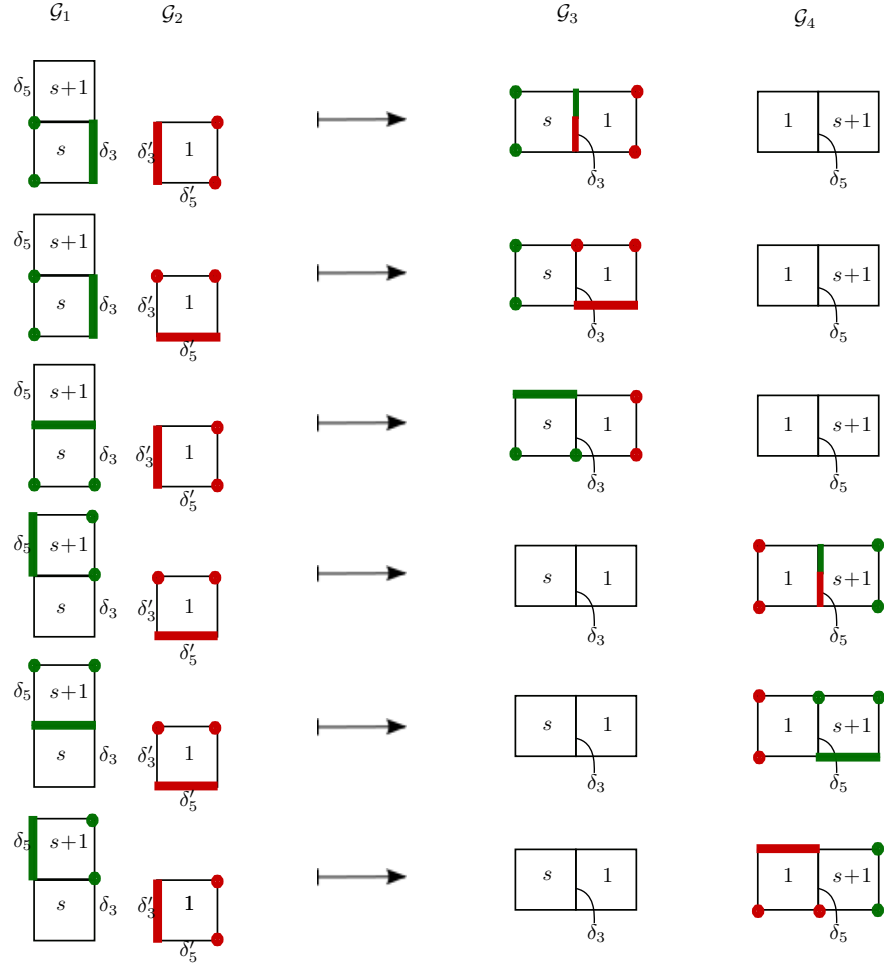


FIGURE 10. The operation ν . The snake graphs $\mathcal{G}_1[s-1, s+1]$ and $\mathcal{G}_2[s'-1, s'+1]$ (left) are straight snake graphs and the overlap is a single tile. The resulting pair $(\nu_{s,1}, \nu_{s,2})$ is shown on the right. Two vertices in two different green circles cannot be matched to each other.

- (i) $\varphi(P_1, P_2) = (P[1, s-1] \cup \sigma_{s,3} \cup P_2[2, d'], P_1[k_4, d])$ if the local configuration of (P_1, P_2) on $(\mathcal{G}_1[s, s+1], \mathcal{G}_2[1, 1])$ is one of the first three cases in Figure 11, and
- (ii) $\varphi(P_1, P_2) = (P_1[s+2, d] \cup \sigma_{s,5} \cup P_2[2, d'], P_1[1, k_6])$ if the local configuration of (P_1, P_2) on $(\mathcal{G}_1[s, s+1], \mathcal{G}_2[1, 1])$ is one of the last three cases in Figure 11.


 FIGURE 11. The operation σ

Here we agree that if $s = d$ then φ is defined by step (i) in the first three cases of Figure 11, where we delete the tiles G , and, in case five of Figure 11, we have $\varphi(P_1, P_2) = (P_2[k_5, d'], P_1[1, k_6])$. In section 7, we prove the theorem by constructing the inverse map of φ . \square

4. SNAKE GRAPHS OF CLUSTER VARIABLES

In this section we recall how snake graphs arise naturally in the theory of cluster algebras. We follow the exposition in [MSW2].

4.1. Cluster algebras. To define a cluster algebra \mathcal{A} we must first fix its ground ring. Let $(\mathbb{P}, \oplus, \cdot)$ be a *semifield*, i.e., an abelian multiplicative group endowed with a binary operation of (*auxiliary*) *addition* \oplus which is commutative, associative, and distributive with respect to the multiplication in \mathbb{P} . The group ring $\mathbb{Z}\mathbb{P}$ will be used as a *ground ring* for \mathcal{A} . One important choice for \mathbb{P} is the tropical semifield; in this case we say that the corresponding cluster algebra is of *geometric type*. Let $\text{Trop}(u_1, \dots, u_m)$ be an abelian group (written multiplicatively) freely generated by the u_j . We define \oplus in $\text{Trop}(u_1, \dots, u_m)$ by

$$(4.1) \quad \prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)},$$

and call $(\text{Trop}(u_1, \dots, u_m), \oplus, \cdot)$ a *tropical semifield*. Note that the group ring of $\text{Trop}(u_1, \dots, u_m)$ is the ring of Laurent polynomials in the variables u_j .

As an *ambient field* for \mathcal{A} , we take a field \mathcal{F} isomorphic to the field of rational functions in n independent variables (here n is the *rank* of \mathcal{A}), with coefficients in $\mathbb{Q}\mathbb{P}$. Note that the definition of \mathcal{F} does not involve the auxiliary addition in \mathbb{P} .

Definition 4.1. A labeled seed in \mathcal{F} is a triple $(\mathbf{x}, \mathbf{y}, B)$, where

- $\mathbf{x} = (x_1, \dots, x_n)$ is an n -tuple from \mathcal{F} forming a free generating set over $\mathbb{Q}\mathbb{P}$,
- $\mathbf{y} = (y_1, \dots, y_n)$ is an n -tuple from \mathbb{P} , and
- $B = (b_{ij})$ is an $n \times n$ integer matrix which is skew-symmetrizable.

That is, x_1, \dots, x_n are algebraically independent over $\mathbb{Q}\mathbb{P}$, and $\mathcal{F} = \mathbb{Q}\mathbb{P}(x_1, \dots, x_n)$. We refer to \mathbf{x} as the (labeled) cluster of a labeled seed $(\mathbf{x}, \mathbf{y}, B)$, to the tuple \mathbf{y} as the coefficient tuple, and to the matrix B as the exchange matrix.

We obtain (unlabeled) seeds from labeled seeds by identifying labeled seeds that differ from each other by simultaneous permutations of the components in \mathbf{x} and \mathbf{y} , and of the rows and columns of B .

We use the notation $[x]_+ = \max(x, 0)$, $[1, n] = \{1, \dots, n\}$, and

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0. \end{cases}$$

Definition 4.2. Let $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed in \mathcal{F} , and let $k \in [1, n]$. The seed mutation μ_k in direction k transforms $(\mathbf{x}, \mathbf{y}, B)$ into the labeled seed $\mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}', \mathbf{y}', B')$ defined as follows:

- The entries of $B' = (b'_{ij})$ are given by

$$(4.2) \quad b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \text{sgn}(b_{ik}) [b_{ik} b_{kj}]_+ & \text{otherwise.} \end{cases}$$

- The coefficient tuple $\mathbf{y}' = (y'_1, \dots, y'_n)$ is given by

$$(4.3) \quad y'_j = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k. \end{cases}$$

- The cluster $\mathbf{x}' = (x'_1, \dots, x'_n)$ is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the exchange relation

$$(4.4) \quad x'_k = \frac{y_k \prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{(y_k \oplus 1)x_k}.$$

We say that two exchange matrices B and B' are *mutation-equivalent* if one can get from B to B' by a sequence of mutations.

Definition 4.3. Consider the n -regular tree \mathbb{T}_n whose edges are labeled by the numbers $1, \dots, n$, so that the n edges emanating from each vertex receive different labels. A cluster pattern is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ to every vertex $t \in \mathbb{T}_n$, such that the seeds assigned to the endpoints of any edge $t \xrightarrow{k} t'$ are obtained from each other by the seed mutation in direction k . The components of Σ_t are written as:

$$(4.5) \quad \mathbf{x}_t = (x_{1;t}, \dots, x_{n;t}), \quad \mathbf{y}_t = (y_{1;t}, \dots, y_{n;t}), \quad B_t = (b_{ij}^t).$$

Clearly, a cluster pattern is uniquely determined by an arbitrary seed.

Definition 4.4. Given a cluster pattern, we denote

$$(4.6) \quad \mathcal{X} = \bigcup_{t \in \mathbb{T}_n} \mathbf{x}_t = \{x_{i,t} : t \in \mathbb{T}_n, 1 \leq i \leq n\},$$

the union of clusters of all the seeds in the pattern. The elements $x_{i,t} \in \mathcal{X}$ are called cluster variables. The cluster algebra \mathcal{A} associated with a given pattern is the \mathbb{ZP} -subalgebra of the ambient field \mathcal{F} generated by all cluster variables: $\mathcal{A} = \mathbb{ZP}[\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$, where $(\mathbf{x}, \mathbf{y}, B)$ is any seed in the underlying cluster pattern.

4.2. Cluster algebras arising from unpunctured surfaces. Let S be a connected oriented 2-dimensional Riemann surface with nonempty boundary, and let M be a nonempty finite subset of the boundary of S , such that each boundary component of S contains at least one point of M . The elements of M are called *marked points*. The pair (S, M) is called a *bordered surface with marked points*.

For technical reasons, we require that (S, M) is not a disk with 1, 2 or 3 marked points.

Definition 4.5. An arc γ in (S, M) is a curve in S , considered up to isotopy, such that:

- (a) the endpoints of γ are in M ;
- (b) γ does not cross itself, except that its endpoints may coincide;
- (c) except for the endpoints, γ is disjoint from the boundary of S ; and
- (d) γ does not cut out a monogon or a bigon.

Curves that connect two marked points and lie entirely on the boundary of S without passing through a third marked point are *boundary segments*. Note that boundary segments are not arcs.

Definition 4.6. For any two arcs γ, γ' in S , let $e(\gamma, \gamma')$ be the minimal number of crossings of arcs α and α' , where α and α' range over all arcs isotopic to γ and γ' , respectively. We say that arcs γ and γ' are compatible if $e(\gamma, \gamma') = 0$.

Definition 4.7. A triangulation is a maximal collection of pairwise compatible arcs (together with all boundary segments).

Definition 4.8. Triangulations are connected to each other by sequences of flips. Each flip replaces a single arc γ in a triangulation T by a (unique) arc $\gamma' \neq \gamma$ that, together with the remaining arcs in T , forms a new triangulation.

Definition 4.9. Choose any triangulation T of (S, M) , and let $\tau_1, \tau_2, \dots, \tau_n$ be the n arcs of T . For any triangle Δ in T , we define a matrix $B^\Delta = (b_{ij}^\Delta)_{1 \leq i \leq n, 1 \leq j \leq n}$ as follows.

- $b_{ij}^\Delta = 1$ and $b_{ji}^\Delta = -1$ if τ_i and τ_j are sides of Δ with τ_j following τ_i in the clockwise order.
- $b_{ij}^\Delta = 0$ otherwise.

Then define the matrix $B_T = (b_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ by $b_{ij} = \sum_{\Delta} b_{ij}^\Delta$, where the sum is taken over all triangles in T .

Note that B_T is skew-symmetric and each entry b_{ij} is either 0, ± 1 , or ± 2 , since every arc τ is in at most two triangles.

Theorem 4.10. [FST, Theorem 7.11] and [FT, Theorem 5.1] Fix a bordered surface (S, M) and let \mathcal{A} be the cluster algebra associated to the signed adjacency matrix of a triangulation. Then the (unlabeled) seeds Σ_T of \mathcal{A} are in bijection with the triangulations T of (S, M) , and the cluster variables are in bijection with the arcs of (S, M) (so we can denote each by x_γ , where γ is an arc). Moreover, each seed in \mathcal{A} is uniquely determined by its cluster. Furthermore, if a triangulation T' is obtained from another triangulation T by flipping an arc $\gamma \in T$ and obtaining γ' , then $\Sigma_{T'}$ is obtained from Σ_T by the seed mutation replacing x_γ by $x_{\gamma'}$.

4.3. Skein relations. In this section we review some results from [MW].

Definition 4.11. A generalized arc in (S, M) is a curve γ in S such that:

- (a) the endpoints of γ are in M ;
- (b) except for the endpoints, γ is disjoint from the boundary of S ; and
- (c) γ does not cut out a monogon or a bigon.

Note that we allow a generalized arc to cross itself a finite number of times. We consider generalized arcs up to isotopy (of immersed arcs).

Definition 4.12. A closed loop in (S, M) is a closed curve γ in S which is disjoint from the boundary of S . We allow a closed loop to have a finite number of self-crossings. As in Definition 4.11, we consider closed loops up to isotopy.

Definition 4.13. A closed loop in (S, M) is called *essential* if it is not contractible and it does not have self-crossings.

Definition 4.14. (Multicurve) We define a multicurve to be a finite multiset of generalized arcs and closed loops such that there are only a finite number of pairwise crossings among the collection. We say that a multicurve is *simple* if there are no pairwise crossings among the collection and no self-crossings.

If a multicurve is not simple, then there are two ways to *resolve* a crossing to obtain a multicurve that no longer contains this crossing and has no additional crossings. This process is known as *smoothing*.

Definition 4.15. (Smoothing) Let γ, γ_1 and γ_2 be generalized arcs or closed loops such that we have one of the following two cases:

- (1) γ_1 crosses γ_2 at a point x ,
- (2) γ has a self-crossing at a point x .

Then we let C be the multicurve $\{\gamma_1, \gamma_2\}$ or $\{\gamma\}$ depending on which of the two cases we are in. We define the smoothing of C at the point x to be the pair of multicurves $C_+ = \{\alpha_1, \alpha_2\}$ (resp. $\{\alpha\}$) and $C_- = \{\beta_1, \beta_2\}$ (resp. $\{\beta\}$).

Here, the multicurve C_+ (resp. C_-) is the same as C except for the local change that replaces the crossing \times with the pair of segments \asymp (resp. \supset).

Since a multicurve may contain only a finite number of crossings, by repeatedly applying smoothings, we can associate to any multicurve a collection of simple multicurves. We call this resulting multiset of multicurves the *smooth resolution* of the multicurve C .

Theorem 4.16. (Skein relations) [MW, Propositions 6.4, 6.5, 6.6] Let C, C_+ , and C_- be as in Definition 4.15. Then we have the following identity in $\mathcal{A}_\bullet(B_T)$,

$$x_C = \pm Y_1 x_{C_+} \pm Y_2 x_{C_-},$$

where Y_1 and Y_2 are monomials in the variables y_{τ_i} . The monomials Y_1 and Y_2 can be expressed using the intersection numbers of the elementary laminations (associated to triangulation T) with the curves in C, C_+ and C_- .

4.4. Snake graphs from surfaces. Let γ be an arc in (S, M) which is not in T . Choose an orientation on γ , let $s \in M$ be its starting point, and let $t \in M$ be its endpoint. We denote by $s = p_0, p_1, p_2, \dots, p_{d+1} = t$ the points of intersection of γ and T in order. Let τ_{i_j} be the arc of T containing p_j , and let Δ_{j-1} and Δ_j be the two triangles in T on either side of τ_{i_j} . Note that each of these triangles has three distinct sides, but not necessarily three distinct vertices, see Figure 12. Let G_j be the graph with 4 vertices and 5 edges, having the shape of a square with a diagonal, such that there is a bijection between the edges of G_j and the 5 arcs in the two triangles Δ_{j-1} and Δ_j , which preserves the signed adjacency of the arcs up to sign and such that the diagonal in G_j corresponds to the arc τ_{i_j} containing the crossing point p_j . Thus G_j is given by the quadrilateral in the triangulation T whose diagonal is τ_{i_j} .

Definition 4.17. Given a planar embedding \tilde{G}_j of G_j , we define the relative orientation $\text{rel}(\tilde{G}_j, T)$ of \tilde{G}_j with respect to T to be ± 1 , based on whether its triangles agree or disagree in orientation with those of T .

For example, in Figure 12, \tilde{G}_j has relative orientation $+1$.

Using the notation above, the arcs τ_{i_j} and $\tau_{i_{j+1}}$ form two edges of a triangle Δ_j in T . Define $\tau_{[j]}$ to be the third arc in this triangle.

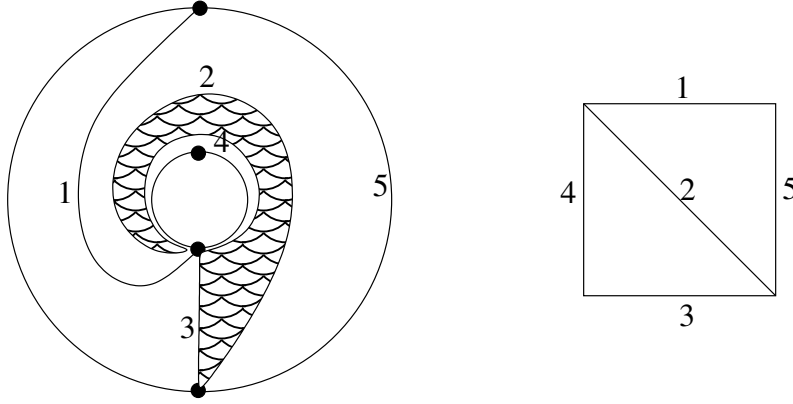


FIGURE 12. On the left, a triangle with two vertices; on the right the tile G_j where $i_j = 2$.

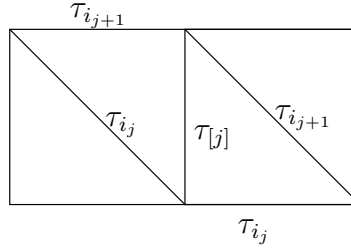


FIGURE 13. Gluing tiles \tilde{G}_j and \tilde{G}_{j+1} along the edge labeled $\tau_{[j]}$

We now recursively glue together the tiles G_1, \dots, G_d in order from 1 to d , so that for two adjacent tiles, we glue G_{j+1} to \tilde{G}_j along the edge labeled $\tau_{[j]}$, choosing a planar embedding \tilde{G}_{j+1} for G_{j+1} so that $\text{rel}(\tilde{G}_{j+1}, T) \neq \text{rel}(\tilde{G}_j, T)$. See Figure 13.

After gluing together the d tiles, we obtain a graph (embedded in the plane), which we denote by $\mathcal{G}_\gamma^\Delta$.

Definition 4.18. The snake graph \mathcal{G}_γ associated to γ is obtained from $\mathcal{G}_\gamma^\Delta$ by removing the diagonal in each tile.

In Figure 14, we give an example of an arc γ and the corresponding snake graph \mathcal{G}_γ . Since γ intersects T five times, \mathcal{G}_γ has five tiles.

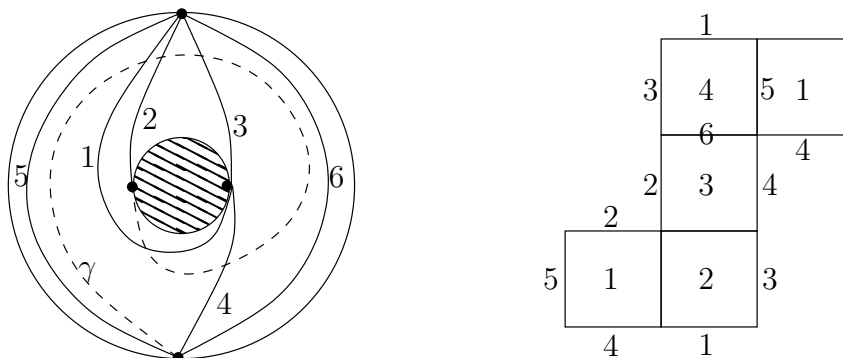
Definition 4.19. If $\tau \in T$ then we define its snake graph \mathcal{G}_τ to be the graph consisting of one single edge with weight x_τ and two distinct endpoints (regardless whether the endpoints of τ are distinct).

4.5. Snake graph formula for cluster variables. Recall that if τ is a boundary segment then $x_\tau = 1$,

Definition 4.20. If γ is a generalized arc and $\tau_{i_1}, \tau_{i_2}, \dots, \tau_{i_d}$ is the sequence of arcs in T which γ crosses, we define the crossing monomial of γ with respect to T to be

$$\text{cross}(T, \gamma) = \prod_{j=1}^d x_{\tau_{i_j}}.$$

Definition 4.21. A perfect matching of a graph \mathcal{G} is a subset P of the edges of \mathcal{G} such that each vertex of \mathcal{G} is incident to exactly one edge of P . If \mathcal{G} is a snake graph and the edges of a perfect matching P of \mathcal{G} are labeled $\tau_{j_1}, \dots, \tau_{j_r}$, then we define the weight $x(P)$ of P to be $x_{\tau_{j_1}} \dots x_{\tau_{j_r}}$.



Definition 4.22. Let γ be a generalized arc. By induction on the number of tiles it is easy to see that the snake graph \mathcal{G}_γ has precisely two perfect matchings which we call the minimal matching $P_- = P_-(\mathcal{G}_\gamma)$ and the maximal matching $P_+ = P_+(\mathcal{G}_\gamma)$, which contain only boundary edges. To distinguish them, if $\text{rel}(\tilde{G}_1, T) = 1$ (respectively, -1), we define e_1 and e_2 to be the two edges of $\mathcal{G}_\gamma^\triangle$ which lie in the counterclockwise (respectively, clockwise) direction from the diagonal of \tilde{G}_1 . Then P_- is defined as the unique matching which contains only boundary edges and does not contain edges e_1 or e_2 . P_+ is the other matching with only boundary edges.

Lemma 4.23. [MS, Theorem 5.1] *The symmetric difference $P_- \ominus P$ is the set of boundary edges of a (possibly disconnected) subgraph \mathcal{G}_P of \mathcal{G}_γ , which is a union of cycles. These cycles enclose a set of tiles $\cup_{j \in J} G_j$, where J is a finite index set.*

$$y(P) = \prod_{j \in J} y_{\tau_{i_j}}.$$

Definition 4.25. Let γ be a generalized arc and let \mathcal{G}_γ be its snake graph.

- (1) If γ has a contractible kink, let $\bar{\gamma}$ denote the corresponding generalized arc with this kink removed, and define $x_\gamma = (-1)x_{\bar{\gamma}}$.
- (2) Otherwise, define

$$x_\gamma = \frac{1}{\text{cross}(T, \gamma)} \sum_P x(P)y(P),$$

Define F_γ^T to be the polynomial obtained from x_γ by specializing all the x_{τ_i} to 1.

Theorem 4.26. [MSW, Thm 4.9] *If γ is an arc, then x_γ is a the cluster variable in \mathcal{A} , written as a Laurent expansion with respect to the seed Σ_T , and F_γ^T is its F-polynomial.*

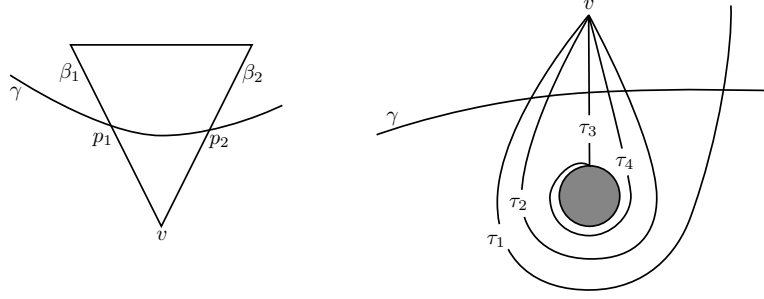


FIGURE 15. Construction of (T, γ) -fans (left). The fan $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$ (right) can not be extended to the right, because the configuration $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6$ does not satisfy condition (3) of Definition 4.27

4.6. Fans. Let T be a triangulation and γ be an arc. Let Δ be a triangle in T with sides β_1, β_2 , and τ , that is crossed by γ in the following way: γ crosses β_1 at the point p_1 and crosses β_2 at the point p_2 , and the segment of γ from p_1 to p_2 lies entirely in Δ , see the left of Figure 15. Then there exists a unique vertex v of the triangle Δ and a unique contractible closed curve ϵ given as the homotopy class of a curve starting at the point v , then following β_1 until the point p_1 , then following γ until the point p_2 and then following β_2 until v . We will use the following notation to describe this definition:

$$\epsilon = v \xrightarrow{\beta_1} p_1 \xrightarrow{\gamma} p_2 \xrightarrow{\beta_2} v .$$

Definition 4.27. A (T, γ) -fan with vertex v is a collection of arcs $\beta_0, \beta_1, \dots, \beta_k$, with $\beta_i \in T$ and $k \geq 0$ with the following properties (see the right of Figure 15):

- (1) γ crosses $\beta_0, \beta_1, \dots, \beta_k$ in order at the points p_0, p_1, \dots, p_k , such that p_i is a crossing point of γ and β_i , and the segment of γ from p_0 to p_k does not have any other crossing points with T ;
- (2) each β_i is incident to v ;
- (3) for each $i < k$, let ϵ_i be the unique contractible closed curve given by

$$v \xrightarrow{\beta_i} p_i \xrightarrow{\gamma} p_{i+1} \xrightarrow{\beta_{i+1}} v ;$$

then for each $i < k - 1$, the concatenation of the curves $\epsilon_i \epsilon_{i+1}$ is homotopic to

$$v \xrightarrow{\beta_i} p_i \xrightarrow{\gamma} p_{i+1} \xrightarrow{\gamma} p_{i+2} \xrightarrow{\beta_{i+2}} v .$$

Condition (3) in the above definition is equivalent to the condition that

$$v \xrightarrow{\beta_i} p_i \xrightarrow{\gamma} p_{i+2} \xrightarrow{\beta_{i+2}} v$$

is contractible.

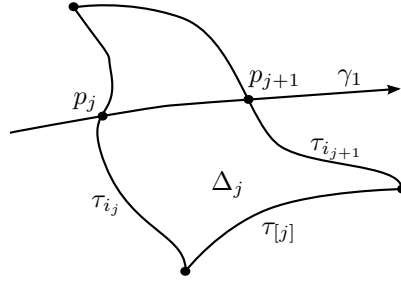
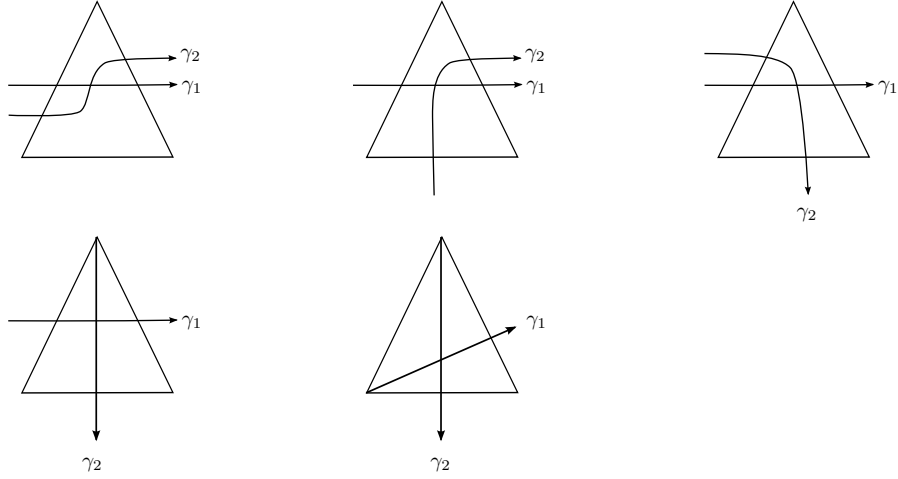
Definition 4.28. A (T, γ) -fan $\beta_0, \beta_1, \dots, \beta_k$ is called maximal if there is no arc $\alpha \in T$ such that $\beta_0, \beta_1, \dots, \beta_k, \alpha$ or $\alpha, \beta_0, \beta_1, \dots, \beta_k$ is a (T, γ) -fan.

Note that if $\tau \in T$ then \mathcal{G}_τ has exactly one perfect matching P and $x(P) = x_\tau$ and $y(P) = 1$.

5. SMOOTHING AND SNAKE GRAPHS

Let (S, M, T) be a triangulated surface. Let γ_1, γ_2 be two arcs in general position in (S, M) which are not in the triangulation T and such that γ_1 crosses γ_2 at a point $p \in S \setminus \partial S$.

Fix an orientation on γ_1 and γ_2 and let p_0, p_1, \dots, p_{d+1} (respectively $p'_0, p'_1, \dots, p'_{d'+1}$) be the crossing points of γ_1 (respectively γ_2) with T in order and including the starting point and the

FIGURE 16. The triangle Δ_j FIGURE 17. Five possible crossings in the triangle Δ_j . In the top left picture, $\tau_{i_j} = \tau'_{i'_j}$ and $\tau_{i_{j+1}} = \tau'_{i'_{j+1}}$, in the top middle picture, $\tau_{i_{j+1}} = \tau'_{i'_{j+1}}$ and, in the top right picture, $\tau_{i_j} = \tau'_{i'_j}$.

endpoint of γ_1 (respectively γ_2). Thus γ_1 runs from p_0 to p_{d+1} and γ_2 from p'_0 to $p'_{d'+1}$. Let Δ_j (respectively $\Delta'_{j'}$) be the triangle in T that contains the segment of γ_1 (respectively γ_2) between the points p_j and p_{j+1} , see Figure 16. Finally, let $\tau_{i_j} \in T$ (respectively $\tau'_{i'_j}$) denote the arc of the triangulation passing through p_j (respectively $p'_{j'}$). Thus τ_{i_j} and $\tau_{i_{j+1}}$ are two sides of the triangle Δ_j . Denote the third side of Δ_j by $\tau_{[j]}$. Similarly the third side of $\Delta'_{j'}$ is denoted by $\tau'_{[j']}$. Without loss of generality, we may assume that the crossing point of γ_1 and γ_2 lies in the interior of some triangle in T and, since γ_1 and γ_2 both run through p , this triangle is Δ_j for some j , and $\Delta'_{j'}$ for some j' .

If Δ_j is the first or the last triangle that γ_1 meets then we assume without loss of generality that Δ_j is the first triangle, that is, $\Delta_j = \Delta_1$. Similarly, if $\Delta'_{j'}$ is the first or the last triangle that γ_2 meets, then we assume without loss of generality that $\Delta'_{j'} = \Delta'_1$. In all other cases, there is at least one side of Δ_j which is crossed both by γ_1 and γ_2 , more precisely, we have $\tau_{i_j} = \tau'_{i'_j}$ or $\tau_{i_j} = \tau'_{i'_{j'+1}}$ or $\tau_{i_{j+1}} = \tau'_{i'_j}$ or $\tau_{i_{j+1}} = \tau'_{i'_{j'+1}}$. We can assume, by changing orientation if necessary, that $\tau_{i_j} = \tau'_{i'_j}$ or $\tau_{i_{j+1}} = \tau'_{i'_{j'+1}}$. Altogether, there are five cases which are illustrated in Figure 17.

Definition 5.1. In the last two cases in Figure 17, we say that γ_1 and γ_2 cross at p with an empty overlap and in the first three cases γ_1 and γ_2 cross at p with a non-empty overlap.

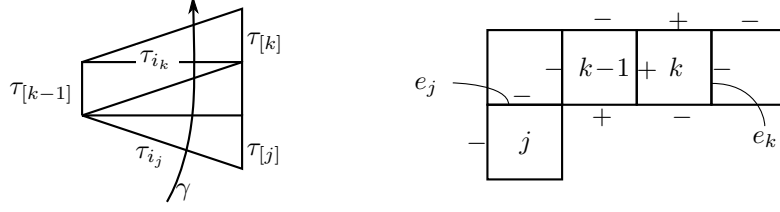


FIGURE 18. A part of a triangulation on the left and the corresponding snake graph on the right. The edges e_j and e_k have the same sign because the arcs $\tau_{[j]}$ and $\tau_{[k]}$ lie on the same side of γ .

Suppose now that γ_1 and γ_2 cross at p with a non-empty local overlap. We want to define the local overlap of γ_1 and γ_2 at p to be the maximal sequence of arcs in the triangulation which are crossed by both γ_1 and γ_2 directly before and after passing through p . Let

$$s = \begin{cases} j+1 & \text{if } \tau_{i_j} \neq \tau'_{i'_j}, \\ j-k & \text{if } \tau_{i_j} = \tau'_{i'_j}, \end{cases}$$

where $k \geq 1$ is the largest integer such that $\tau_{i_{j-1}} = \tau'_{i'_{j-1}}, \dots, \tau_{i_{j-k}} = \tau'_{i'_{j-k}}$ and let

$$t = \begin{cases} j & \text{if } \tau_{i_{j+1}} \neq \tau'_{i'_{j+1}}, \\ j+\ell & \text{if } \tau_{i_{j+1}} = \tau'_{i'_{j+1}}, \end{cases}$$

where $\ell \geq 1$ is the largest integer such that $\tau_{i_{j+1}} = \tau'_{i'_{j+1}}, \dots, \tau_{i_{j+\ell}} = \tau'_{i'_{j+\ell}}$.

Definition 5.2. We call the sequence $(\tau_{i_s}, \tau_{i_{s+1}}, \dots, \tau_{i_t}) = (\tau'_{i'_s}, \tau'_{i'_{s+1}}, \dots, \tau'_{i'_t})$ the local overlap of γ_1 and γ_2 at p .

5.1. Crossing arcs and crossing snake graphs.

Theorem 5.3. Let γ_1 and γ_2 be two arcs and \mathcal{G}_1 and \mathcal{G}_2 their corresponding snake graphs. Then γ_1 and γ_2 cross with a non-empty local overlap $(\tau_{i_s}, \dots, \tau_{i_t}) = (\tau'_{i'_s}, \dots, \tau'_{i'_t})$ if and only if \mathcal{G}_1 and \mathcal{G}_2 cross in $\mathcal{G}_1[s, t] \cong \mathcal{G}_2[s', t']$.

Proof. Choose a sign function f on the overlap of the snake graphs \mathcal{G}_1 and \mathcal{G}_2 and let f_1 and f_2 be the induced sign functions on \mathcal{G}_1 and \mathcal{G}_2 , respectively. As usual, let e_1, \dots, e_{d-1} (respectively $e'_1, \dots, e'_{d'-1}$) be the interior edges of \mathcal{G}_1 (respectively \mathcal{G}_2). Recall that the edge e_j (respectively $e'_{j'}$) corresponds to the arc $\tau_{[j]}$ (respectively $\tau'_{[j']}$) of the triangle Δ_j (respectively $\Delta'_{j'}$) in the triangulation T .

A zigzag in the snake graph corresponds to a fan in the triangulation, and a straight subgraph of the snake graph corresponds to a zigzag on the triangulation. Therefore two edges e_j, e_k (respectively $e'_{j'}, e'_{k'}$) of the snake graph have the same sign with respect to f_1 (respectively f_2) if and only if the corresponding arcs $\tau_{[j]}, \tau_{[k]}$ (respectively $\tau'_{[j]}, \tau'_{[k]}$) lie on the same side of γ_1 (respectively γ_2), see Figure 18.

Suppose first that $s > 1$ and $t < d$. Then we have one of the two situations shown in Figure 19.

In both cases γ_2 crosses $\tau_{i_s}, \dots, \tau_{i_t}$ but does not cross $\tau_{i_{s-1}}$ nor $\tau_{i_{t+1}}$. Thus in the first case γ_2 cannot cross γ_1 in the overlap. In the second case, γ_2 must cross γ_1 in the overlap, since the arcs $\tau_{[s-1]}$ and $\tau_{[t]}$ as well as their endpoints lie on opposite sites of γ_1 relative to the overlap. Therefore γ_1 and γ_2 cross if and only if $f_1(e_{s-1}) = -f_1(e_t)$. This implies that the snake graphs \mathcal{G}_1 and \mathcal{G}_2 cross, by Definition 2.1.

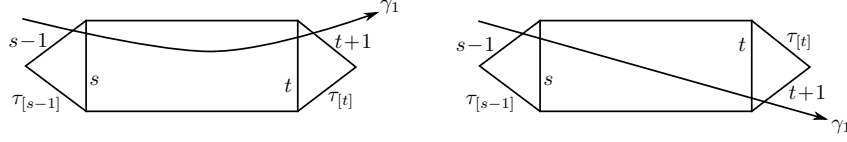


FIGURE 19. Proof of Theorem 5.3; $f_1(e_{s-1}) = f_1(e_t)$ on the left, and $f_1(e_{s-1}) = -f_1(e_t)$ on the right.

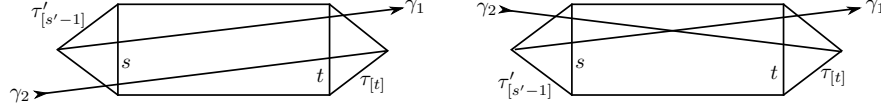


FIGURE 20. Proof of Theorem 5.3

Suppose now that $s = 1$, $t < d$, $s' > 1$ and $t' = d'$. Then we have one of the two situations shown in Figure 20.

In the first case, γ_1 and γ_2 do not cross in the overlap and $f_1(e_t) = -f_2(e'_{s'-1})$ because the arcs $\tau_{[t]}$ and $\tau'_{[s'-1]}$ lie on opposite sides of the arcs γ_1 and γ_2 . In the second case γ_1 and γ_2 cross in the overlap and $f_1(e_t) = f_2(e'_{s'-1})$. Thus by Definition 2.1, γ_1 and γ_2 cross if and only if \mathcal{G}_1 and \mathcal{G}_2 cross.

By symmetry, this proves the statement. \square

If γ is an arc which is not in the triangulation, then the segment of γ from its starting point to its first crossing with the triangulation is called the *initial segment* of γ .

Theorem 5.4. *Let γ_1 and γ_2 be two arcs and \mathcal{G}_1 and \mathcal{G}_2 their associated snake graphs. Suppose that the first tile of \mathcal{G}_2 is given by*

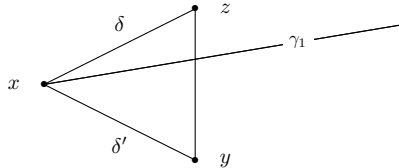
$$\begin{array}{c} \delta \quad \boxed{k} \\ \delta' \end{array}$$

Then γ_1 crosses the initial segment of γ_2 if and only if one of the following hold.

- *The first or last tile of \mathcal{G}_1 is G_δ , or*
- *the first or last tile of \mathcal{G}_1 is $G_{\delta'}$, or*
- *\mathcal{G}_1 contains one of the following snake graphs*

$$\begin{array}{|c|c|c|} \hline \delta & k & \delta' \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline \delta' & k & \delta \\ \hline \end{array}$$

Proof. The configuration of the first tile of \mathcal{G}_2 translates into the following picture.



Thus γ_1 crosses the initial segment of γ_2 if and only if γ_1 crosses δ and ends at y or γ_1 crosses δ' and ends at z or γ_1 crosses δ and δ' . This translates to the three cases in the statement. \square

5.2. Smoothing arcs and resolving snake graphs.

Theorem 5.5. *Let γ_1 and γ_2 be two arcs which cross with a non-empty local overlap, and let \mathcal{G}_1 and \mathcal{G}_2 be the corresponding snake graphs with overlap \mathcal{G} . Then the snake graphs of the four*

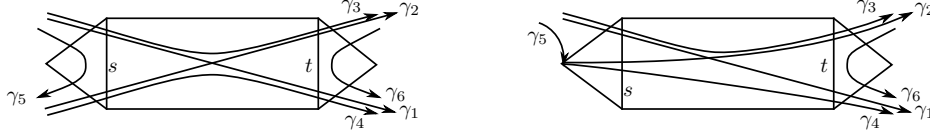


FIGURE 21. Proof of Theorem 5.5

arcs obtained by smoothing the crossing of γ_1 and γ_2 in the overlap are given by the resolution $\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$ of the crossing of the snake graphs \mathcal{G}_1 and \mathcal{G}_2 at the overlap \mathcal{G} .

Remark 5.6. We do not assume that γ_1 and γ_2 cross only once. If the arcs cross multiple times the theorem can be used to resolve any of the crossings.

Remark 5.7. In [MSW2], the authors considered also smoothing of generalized arcs (allowing self-crossings) and essential loops. The description of the resolutions of the corresponding snake graphs will be given in a forthcoming paper.

Proof. As usual, let $(\tau_{i_s}, \dots, \tau_{i_t}) = (\tau'_{i'_s}, \dots, \tau'_{i'_t})$ denote the local overlap of γ_1 and γ_2 at the crossing under consideration. Then the four arcs obtained by smoothing the crossing are represented by

$$\begin{aligned} \gamma_3 &= \gamma_{1,1} \cdot \gamma_{2,2} & \gamma_5 &= \gamma_{1,1} \cdot \bar{\gamma}_{2,1} \\ \gamma_4 &= \gamma_{2,1} \cdot \gamma_{1,2} & \gamma_6 &= \bar{\gamma}_{2,2} \cdot \gamma_{1,2} \end{aligned}$$

(see Figure 21) where $\alpha \cdot \beta$ denotes the concatenation of the paths α and β and $\bar{\alpha}$ denotes the path α with the opposite orientation, $\gamma_{i,1}$ denotes the segment of the arc γ_i from its starting point to the crossing point p and $\gamma_{i,2}$ the segment from p to the terminal point, for $i = 1, 2$. Recall that arcs are defined up to homotopy, so for example the arc γ_5 does not cross τ_{i_s} .

Suppose first that $s > 1$, $s' > 1$, $t < d$, and $t' < d'$. From the construction it is obvious that the snake graphs \mathcal{G}_i of the arcs γ_i , for $i = 3, 4, 5, 6$, are given exactly by the four snake graphs of $\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$ in Definition 2.3.

Now suppose that we do not have $s > 1$, $s' > 1$, $t < d$, and $t' < d'$. Observe that that we cannot have $s = 1$ and $s' = 1$ simultaneously because γ_1 and γ_2 cross in the overlap, and, similarly, we cannot have $t = d$ and $t' = d'$ simultaneously. Therefore we have to consider the cases where one of the two arcs starts or ends right before or right after the overlap. Since these cases are all symmetric, we may assume without loss of generality that $s' = 1$ and $s > 1$, $t < d$ and $t' < d'$. This case is illustrated on the right side of Figure 21. In this case, the snake graphs \mathcal{G}_3 , \mathcal{G}_4 and \mathcal{G}_6 are the same as in the first case, but \mathcal{G}_5 now is of the form $\mathcal{G}_5 = \mathcal{G}_1[1, k]$, where $1 \leq k < s - 1$ and k is the largest integer such that τ_{i_k} is not in the maximal (T, γ_1) -fan ending at $\tau_{i_{s-1}}$.

Indeed, this follows because every arc in this fan crosses γ_1 but not γ_5 , since each arc in the fan ends at the endpoint of γ_5 . Since fans in the triangulation correspond to zigzag subgraphs of the snake graph, we see from the definition of sign functions that any sign function f_1 on \mathcal{G}_1 satisfies $f_1(e_{s-1}) = f_1(e_\ell)$ for each $k < \ell \leq s - 1$ and $f_1(e_{s-1}) = -f_1(e_k)$. Therefore \mathcal{G}_5 is exactly as in the definition of $\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$. \square

So far, we have considered two arcs which cross with a non-empty local overlap. Now we study two arcs which cross with an empty local overlap.

Theorem 5.8. *Let γ_1 and γ_2 be two arcs which cross in a triangle Δ with an empty local overlap, and let \mathcal{G}_1 and \mathcal{G}_2 be the corresponding snake graphs. Assume $\Delta = \Delta'_0$ is the first triangle γ_2 meets. Then the snake graphs of the four arcs obtained by smoothing the crossing of γ_1 and γ_2 in Δ are given by the resolution $\text{Graft}_{s, \delta_3}(\mathcal{G}_1, \mathcal{G}_2)$ of the grafting of \mathcal{G}_2 on \mathcal{G}_1 in G_s , where $0 \leq s \leq d$ is such that $\Delta = \Delta_s$ and if $s = 0$ or $s = d$ then δ_3 is the unique side of Δ that is not crossed by neither γ_1 nor γ_2 .*

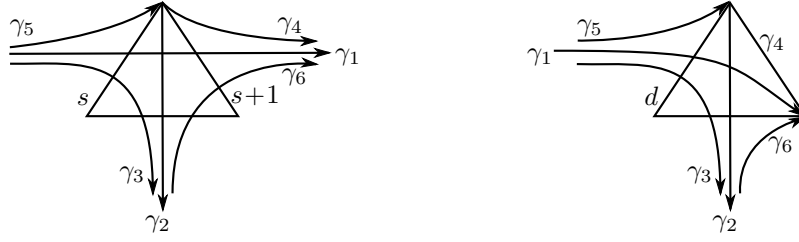


FIGURE 22. Proof of Theorem 5.8

Proof. As before, let $\gamma_{i,1}$ denote the segment of γ_i from the starting point until the crossing point, and let $\gamma_{i,2}$ denote the segment from the crossing point to the terminal point. Then the four arcs obtained by smoothing are represented by

$$\begin{aligned}\gamma_3 &= \gamma_{1,1} \cdot \gamma_{2,2} & \gamma_5 &= \gamma_{1,1} \cdot \bar{\gamma}_{2,1} \\ \gamma_4 &= \gamma_{2,1} \cdot \gamma_{1,2} & \gamma_6 &= \bar{\gamma}_{2,2} \cdot \gamma_{1,2}\end{aligned}$$

where $\alpha \cdot \beta$ denotes the concatenation of the paths α and β and $\bar{\alpha}$ is the path α with the opposite orientation.

Suppose first that $\Delta = \Delta_s$ with $0 < s < d$. Thus γ_1 crosses two sides τ_{i_s} and $\tau_{i_{s+1}}$ of Δ and γ_2 crosses the third side $\tau_{i'_1}$ of Δ , see the left picture in Figure 22.

Let \mathcal{F} denote the maximal (T, γ_1) -fan containing the arcs τ_{i_s} and $\tau_{i_{s+1}}$ and determined by the crossing. Then the snake graphs \mathcal{G}_i of the arcs γ_i , for $i = 3, 4, 5, 6$ are given as follows.

$$\begin{aligned}\mathcal{G}_3 &= \mathcal{G}_1[1, s] \cup \mathcal{G}_2 && \text{glued along the edge of } G_s \text{ and } G'_1 \text{ labeled } \tau_{i_{s+1}} && ; \\ \mathcal{G}_4 &= \begin{cases} \mathcal{G}_1[k_4, d] & \text{where } k_4 > s+1 \text{ is the least integer such that } \tau_{i_{k_4}} \notin \mathcal{F}, \text{ if such a } k_4 \text{ exists;} \\ \{\gamma_4\} & \text{otherwise;} \end{cases} \\ \mathcal{G}_5 &= \begin{cases} \mathcal{G}_1[1, k_5] & \text{where } 1 \leq k_5 < s \text{ is the largest integer such that } \tau_{i_{k_5}} \notin \mathcal{F}, \text{ if such a } k_5 \text{ exists;} \\ \{\gamma_5\} & \text{otherwise;} \end{cases} \\ \mathcal{G}_6 &= \bar{\mathcal{G}}_2 \cup \mathcal{G}_1[s+1, d] && \text{glued along the edge of } G_{s+1} \text{ and } G'_1 \text{ labeled } \tau_{i_s} && ;\end{aligned}$$

Note that if the k_4 in the description of \mathcal{G}_4 does not exist, then γ_4 itself must be in the fan \mathcal{F} , hence $\gamma_4 \in T$ and $\mathcal{G}_4 = \{\gamma_4\}$. Similarly, if the k_5 in the description of \mathcal{G}_5 does not exist, then $\mathcal{G}_5 = \{\gamma_5\}$. Thus these are exactly the snake graphs in $\text{Graft}_{s, \delta_3}(\mathcal{G}_1, \mathcal{G}_2)$.

Now suppose that $\Delta = \Delta_d$ is the last triangle γ_1 meets, see the right side of Figure 22. Then

$$\begin{aligned}\gamma_3 &= \gamma_{1,1} \cdot \gamma_{2,2} & \gamma_5 &= \gamma_{1,1} \cdot \bar{\gamma}_{2,1} \\ \gamma_4 &\in T & \gamma_6 &= \bar{\gamma}_{2,2} \cdot \gamma_{1,2}\end{aligned}$$

and γ_4 is the unique side of the triangle Δ that is not crossed by γ_1 or γ_2 . Let \mathcal{F}_1 be the maximal (T, γ_1) -fan containing τ_{i_d} and γ_4 , and let \mathcal{F}_2 be the maximal (T, γ_2) -fan containing $\tau'_{i'_1}$ and γ_4 . Then the snake graphs of the arcs γ_i for $i = 3, 4, 5, 6$ are the following.

$$\begin{aligned}
\mathcal{G}_3 &= \mathcal{G}_1 \cup \mathcal{G}_2 && \text{glued along the edge } G_d \text{ and } G'_1 \text{ labeled } \gamma_4 && ; \\
\mathcal{G}_4 &= \{\gamma_4\} \\
\mathcal{G}_5 &= \begin{cases} \mathcal{G}_1[1, k_5] & \text{where } 1 \leq k_5 < d \text{ is the largest integer such that} \\ & \tau_{ik_5} \notin \mathcal{F}_1 \text{ if such a } k_5 \text{ exists;} \\ \{\gamma_5\} & \text{otherwise;} \end{cases} \\
\mathcal{G}_6 &= \begin{cases} \bar{\mathcal{G}}_1[1, k_6] & \text{where } 1 \leq k_6 < d \text{ and } k_6 \text{ is the largest integer such} \\ & \text{that } \tau'_{i'_{k_6}} \notin \mathcal{F}_2 \text{ if such a } k_6 \text{ exists;} \\ \{\gamma_6\} & \text{otherwise;} \end{cases}
\end{aligned}$$

Since fans in the triangulation correspond to zigzag subgraphs in the snake graph, we see from the definition of sign functions that these snake graphs are exactly the ones in the definition of $\text{Graft}_{s, \delta_3}(\mathcal{G}_1, \mathcal{G}_2)$ with $\delta_3 = \gamma_4$.

The case $\Delta = \Delta_0$ is similar to the case $\Delta = \Delta_d$ and therefore left to the reader. \square

6. PRODUCTS OF CLUSTER VARIABLES

Let $\mathcal{A} = \mathcal{A}(S, M, T)$ be the cluster algebra associated to the surface (S, M) with principal coefficients in the initial seed $\sum_T = (\mathbf{x}_T, \mathbf{y}_T, Q_T)$ at the triangulation $T = \{\tau_1, \tau_2, \dots, \tau_n\}$ where

$$\begin{aligned}
\mathbf{x} &= (x_i | i = 1, \dots, n, \tau_i \in T) \\
\mathbf{y} &= (y_i | i = 1, \dots, n, \tau_i \in T)
\end{aligned}$$

6.1. Non-empty overlaps. If \mathcal{G} is a snake graph associated to an arc γ in a triangulated surface (S, M, T) then each tile of \mathcal{G} corresponds to a quadrilateral in the triangulation T , and we denote by $\tau_{i(G)} \in T$ the diagonal of that quadrilateral. With this notation we define

$$\begin{aligned}
x(\mathcal{G}) &= \prod_{G \text{ tile in } \mathcal{G}} x_{i(G)} \\
y(\mathcal{G}) &= \prod_{G \text{ tile in } \mathcal{G}} y_{i(G)}
\end{aligned}$$

If $\mathcal{G} = \{\tau\}$ consists of a single edge, we let $x(\mathcal{G}) = 1$ and $y(\mathcal{G}) = 1$.

Let γ_1 and γ_2 be two arcs which cross with a non-empty local overlap. Let x_{γ_1} and x_{γ_2} be the corresponding cluster variables and \mathcal{G}_1 and \mathcal{G}_2 the snake graphs with corresponding overlap \mathcal{G} . Recall that $\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$ consists of two pairs $(\mathcal{G}_3, \mathcal{G}_4)$ and $(\mathcal{G}_5, \mathcal{G}_6)$ of snake graphs.

Define the *closure* $\tilde{\mathcal{G}}$ of the overlap \mathcal{G} to be the union of all tiles in $\mathcal{G}_1 \cup \mathcal{G}_2$ which are not in $\mathcal{G}_5 \cup \mathcal{G}_6$. Let $\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$ be the resolution of the crossing of \mathcal{G}_1 and \mathcal{G}_2 at \mathcal{G} , and define the *Laurent polynomial of the resolution* by

$$\mathcal{L}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)) = \mathcal{L}(\mathcal{G}_3 \sqcup \mathcal{G}_4) + y(\tilde{\mathcal{G}})\mathcal{L}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$$

where

$$\mathcal{L}(\mathcal{G}_k \sqcup \mathcal{G}_\ell) = \frac{1}{x(\mathcal{G}_k)x(\mathcal{G}_\ell)} \sum_{P \in \text{Match}(\mathcal{G}_k \sqcup \mathcal{G}_\ell)} x(P)y(P)$$

Theorem 6.1. *Let γ_1 and γ_2 be two arcs which cross with a non-empty local overlap and let \mathcal{G}_1 and \mathcal{G}_2 be the corresponding snake graphs with local overlap \mathcal{G} . Then*

$$\mathcal{L}(\mathcal{G}_1 \sqcup \mathcal{G}_2) = \mathcal{L}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2))$$

Proof. First we note that

$$(6.1) \quad x(\mathcal{G}_1 \sqcup \mathcal{G}_2) = x(\mathcal{G}_3 \sqcup \mathcal{G}_4) = x(\mathcal{G}_5 \sqcup \mathcal{G}_6)x(\tilde{\mathcal{G}})x(\mathcal{G}),$$

where the first identity holds because $\mathcal{G}_1 \sqcup \mathcal{G}_2$ and $\mathcal{G}_3 \sqcup \mathcal{G}_4$ have the same set of tiles, and the second identity holds because $\tilde{\mathcal{G}}$ consists of the tiles of $(\mathcal{G}_1 \cup \mathcal{G}_2) \setminus (\mathcal{G}_5 \cup \mathcal{G}_6)$ and the overlap \mathcal{G} consists of the tiles that appear in both \mathcal{G}_1 and \mathcal{G}_2 .

Using the equation (6.1) on the definition of $\mathcal{L}(\mathcal{G}_1 \sqcup \mathcal{G}_2)$ as well as the bijection $\varphi = (\varphi_{34}, \varphi_{56}) : \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2) \rightarrow \text{Match Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$ of Theorem 3.1, we obtain

$$\begin{aligned} \mathcal{L}(\mathcal{G}_1 \sqcup \mathcal{G}_2) &= \frac{1}{x(\mathcal{G}_3 \sqcup \mathcal{G}_4)} \sum_{\varphi_{34}(P) \in \text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4)} x(P)y(P) \\ &\quad + \frac{1}{x(\mathcal{G})x(\tilde{\mathcal{G}})x(\mathcal{G}_5 \sqcup \mathcal{G}_6)} \sum_{\varphi_{56}(P) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)} x(P)y(P). \end{aligned}$$

On the other hand,

$$\mathcal{L}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)) = \mathcal{L}(\mathcal{G}_3 \sqcup \mathcal{G}_4) + y(\tilde{\mathcal{G}})\mathcal{L}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$$

and therefore it suffices to show the following lemma. \square

Lemma 6.2. *Let $P \in \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2)$.*

- (a) $x(\varphi_{34}(P)) y(\varphi_{34}(P)) = x(P)y(P)$ if $\varphi(P) \in \text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4)$.
- (b) $x(\varphi_{56}(P)) y(\varphi_{56}(P)) = x(P)y(P)/x(\tilde{\mathcal{G}})x(\mathcal{G})y(\tilde{\mathcal{G}})$ if $\varphi(P) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$.

Proof. Let $P \in \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2)$ and let P_i be its restriction to \mathcal{G}_i , $i = 1, 2$.

(a) Since the mapping $P \rightarrow \varphi_{34}(P)$ preserves the weight of the edges on the matching, we have $x(\varphi_{34}(P)) = x(P)$. The identity $y(\varphi_{34}(P)) = y(P)$ follows from an inspection of the eight cases in Figure 6 and the four cases of Figure 8.

(b) By definition

$$(6.2) \quad \varphi_{56}(P) = P|_{\mathcal{G}_5 \sqcup \mathcal{G}_6} \setminus \{ \text{glueing edges} \}$$

where the glueing edge for \mathcal{G}_5 has weight x_{i_s} if $s > 1$ and $s' > 1$, and in all other cases \mathcal{G}_5 has no glueing edge, and the glueing edge for \mathcal{G}_6 has weight x_{i_t} if $t < d$ and $t' < d'$, and in all other cases \mathcal{G}_6 has no glueing edge. Since the glueing edges are in the interior of \mathcal{G}_5 and \mathcal{G}_6 , removing them from the matching does not change the y -monomials. Thus

$$y(\varphi_{56}(P)) = y(P)/y(\tilde{\mathcal{G}})$$

It remains to study the x -monomials. First note that if P_1 or P_2 contains an interior edge of $\tilde{\mathcal{G}}$ then $\varphi(P)$ would be in $\text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4)$. Thus in our situation, we must have that on $\tilde{\mathcal{G}}$ both matchings P_1 and P_2 consist of boundary edges of $\tilde{\mathcal{G}}$. Moreover, since $\varphi(P) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$, it follows that P_1 and P_2 must be as in Figure 7 and Figure 9 for all tiles in the overlap. In particular, P_1 and P_2 do not share an edge on $\tilde{\mathcal{G}}$.

Therefore equation (6.2) implies that if $s > 1, s' > 1, t < d$ and $t' < d'$ then

$$(6.3) \quad x(\varphi_{56}(P)) = \frac{x(P|_{\mathcal{G}_5 \sqcup \mathcal{G}_6})}{x_{i_s}x_{i_t}} = \frac{x(P)}{x_{i_s}x_{i_t}} \frac{x_{i_{s-1}}x_{i'_{s'-1}}x_{i_{t+1}}x_{i'_{t'+1}}}{\prod_{\tau \in \partial \mathcal{G}} x_{\tau}}$$

where the product runs over all boundary edges of \mathcal{G} . Indeed, in this case $\tilde{\mathcal{G}} = \mathcal{G}$ and the four variables $x_{i_{s-1}}, x_{i'_{s'-1}}, x_{i_{t+1}}$ and $x_{i'_{t'+1}}$ appear in the numerator, because these are the weights of the four edges which connect \mathcal{G} with $\mathcal{G}_1 \setminus \mathcal{G}$ and with $\mathcal{G}_2 \setminus \mathcal{G}$. These edges are boundary edges in \mathcal{G} but interior edges in \mathcal{G}_1 , respectively \mathcal{G}_2 .

To finish the proof in the case where $s > 1, s' > 1, t < d$, and $t' < d'$ it suffices to show that

$$(6.4) \quad x(\mathcal{G})x(\tilde{\mathcal{G}}) = \frac{x_{i_s}x_{i_t}}{x_{i_{s-1}}x_{i'_{s'-1}}x_{i_{t+1}}x_{i'_{t'+1}}} \prod_{\tau \in \partial \mathcal{G}} x_{\tau}$$

We proceed by induction on $\ell = t - s + 1$ the number of tiles in \mathcal{G} . If $\ell = 1$, then \mathcal{G} consists of a single tile with boundary weights $x_{i_{s-1}}, x_{i'_{s'-1}}, x_{i_{t+1}}, x_{i'_{t'+1}}$. Moreover, $s = t$, and $\mathcal{G} = \tilde{\mathcal{G}}$, since $s > 1, s' > 1, t < d$, and $t' < d'$. This proves equation (6.4) in this case.

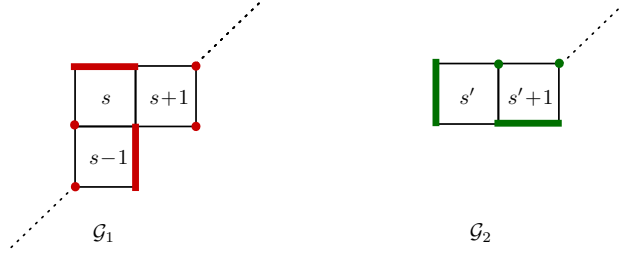


FIGURE 23. Proof of Lemma 6.2

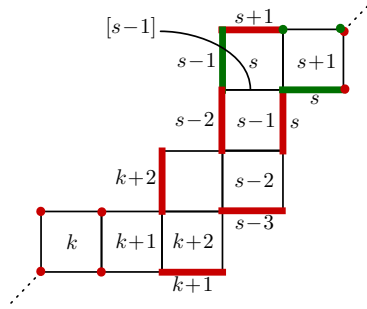


FIGURE 24. Proof of Lemma 6.2

Suppose now that $\ell > 1$. Then $x(\tilde{\mathcal{G}}) = x(\mathcal{G}) = x_{i_s} x_{i_{s+1}} \dots x_{i_t}$. Moreover, except for the four extremal edges with weights $x_{i_{s-1}}$, $x_{i'_{s'-1}}$, $x_{i_{t+1}}$, $x_{i'_{t'+1}}$, every boundary edge of \mathcal{G} has weight $x_{i_s}, x_{i_{s+1}}, \dots, x_{i_{t-1}}$ or x_{i_t} , each of the weights $x_{i_{s+1}}, \dots, x_{i_{t-1}}$ appears exactly twice and the weights x_{i_s} and x_{i_t} appears exactly once in the boundary of \mathcal{G} . Thus

$$\prod_{\tau \in \partial \mathcal{G}} x_\tau = x_{i_{s+1}}^2 \cdots x_{i_{t-1}}^2 x_{i_s} x_{i_t} x_{i_{s-1}} x_{i'_{s'-1}} x_{i_{t+1}} x_{i'_{t'+1}}$$

which shows (6.4) in this case.

Now suppose $s' = 1$. Then $s > 1$, because γ and γ' cross in the overlap. Suppose first that $t < d$ and $t' < d'$. Then it follows from the definition of φ that the matching $P = (P_1, P_2)$ must be of the form shown in Figure 7, but with the tile $G'_{s'-1}$ removed. Only the third case in Figure 7 defines a matching after removing $G'_{s'-1}$, thus $P = (P_1, P_2)$ is of the form shown in Figure 23.

Let $k < s - 1$ be the largest integer such that the sign function of \mathcal{G}_1 agrees on the edges e_k and e_{s-1} (if such a k exists) as in the definition of \mathcal{G}_5 . Then we have a situation similar to the example shown in Figure 24, where $k = s - 6$. The red edges are in P_1 , and the green edges are in P_2 . The tile G_k is matched by $\varphi_{56}(P)$. Therefore

$$(6.5) \quad x(\varphi_{56}(P)) = \frac{x(P) x_{[s-1]} x_{i_{s-1}} x_{i_{t+1}} x_{i'_{t'+1}}}{x_{i_t} \left(\prod_{j=k+1}^s x_{i_j} \right) \left(\prod_{\tau \in \partial \mathcal{G}} x_\tau \right)}.$$

Note that $x_{i_{s-1}}$ appears in both products in the denominator and once in the numerator. To finish the proof in this case, it suffices to show that

$$(6.6) \quad x(\mathcal{G}) x(\tilde{\mathcal{G}}) = \frac{x_{i_t}}{x_{[s-1]} x_{i_{s-1}} x_{i_{t+1}} x_{i'_{t'+1}}} \left(\prod_{j=k+1}^s x_{i_j} \right) \left(\prod_{\tau \in \partial \mathcal{G}} x_\tau \right).$$

Again we proceed by induction on $\ell = t - s + 1$. If $\ell = 1$ then \mathcal{G} consists of a single tile and the statement follows simply by computing

$$\begin{aligned} \prod_{\tau \in \partial \mathcal{G}} x_\tau &= x_{[s-1]} x_{i_{s-1}} x_{i_{t+1}} x_{i'_{t'+1}} \\ x(\mathcal{G}) &= x_{i_s} = x_{i_t} & x(\tilde{\mathcal{G}}) &= \prod_{j=k+1}^s x_{i_j}. \end{aligned}$$

Suppose now that $\ell > 1$. Then

$$x(\mathcal{G})x(\tilde{\mathcal{G}}) = \left(\prod_{j=s}^t x_{i_j} \right) \left(\prod_{j=k+1}^t x_{i_j} \right) = \left(\prod_{j=k+1}^s x_{i_j} \right) x_{i_s} \left(\prod_{j=s+1}^t x_{i_j}^2 \right)$$

On the other hand, except for the four extremal edges with weights $x_{[s-1]}$, $x_{i_{s-1}}$, $x_{i_{t+1}}$, and $x_{i'_{t'+1}}$, every boundary edge of \mathcal{G} has weight $x_{i_s}, x_{i_{s+1}}, \dots, x_{i_{t-1}}$ or x_{i_t} , the weights $x_{i_{s+1}}, \dots, x_{i_{t-1}}$ appear exactly twice and x_{i_s} and x_{i_t} exactly once. Thus

$$\prod_{\tau \in \partial \mathcal{G}} x_\tau = \left(\prod_{j=s+1}^{t-1} x_{i_j}^2 \right) x_{i_s} x_{i_t} x_{[s-1]} x_{i_{s-1}} x_{i_{t+1}} x_{i'_{t'+1}}$$

and this shows equation (6.6). The cases where $s = 1, t = d$ or $t' = d'$ are similar. \square

6.2. Empty overlaps. Now let γ_1 and γ_2 be two arcs which cross in a triangle Δ with an empty overlap. We may assume without loss of generality that Δ is the first triangle γ_2 meets. Let x_{γ_1} and x_{γ_2} be the corresponding cluster variables and \mathcal{G}_1 and \mathcal{G}_2 be their associated snake graphs, respectively. We know from Theorem 5.8 that the snake graphs of the arcs obtained by smoothing the crossing of γ_1 and γ_2 are given by the resolution $\text{Graft}_{s, \delta_3}(\mathcal{G}_1, \mathcal{G}_2)$ of the grafting of \mathcal{G}_2 on \mathcal{G}_1 in G_s , where s is such that $\Delta = \Delta_s$ and, if $s = 0$, then δ_3 is the unique side of Δ which is not crossed neither by γ_1 nor γ_2 .

The edge of G_s which is the glueing edge for the grafting is called the *grafting edge*. We say that the grafting edge is *minimal* if it belongs to the minimal matching on \mathcal{G}_1 .

Recall that $\text{Graft}_{s, \delta_3}(\mathcal{G}_1, \mathcal{G}_2)$ is a pair $(\mathcal{G}_3 \sqcup \mathcal{G}_4), (\mathcal{G}_5 \sqcup \mathcal{G}_6)$. Let

$$\mathcal{L}(\text{Graft}_{s, \delta_3}(\mathcal{G}_1, \mathcal{G}_2)) = y_{34} \mathcal{L}(\mathcal{G}_3 \sqcup \mathcal{G}_4) + y_{56} \mathcal{L}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$$

where

$$(6.7) \quad \begin{cases} y_{34} = 1 & \text{and} & y_{56} = \prod_{j=k_5+1}^{s-1} y_{i_j} & \text{if the grafting edge is minimal;} \\ y_{34} = \prod_{j=s+1}^{k_4-1} y_{i_j} & \text{and} & y_{56} = 1 & \text{otherwise.} \end{cases}$$

Theorem 6.3. *With the notation above, we have*

$$\mathcal{L}(\mathcal{G}_1 \sqcup \mathcal{G}_2) = \mathcal{L}(\text{Graft}_{s, \delta_3}(\mathcal{G}_1, \mathcal{G}_2))$$

Proof. The cases $s = d$ and $s = 0$ are symmetric and therefore we consider only the case $0 < s \leq d$. By definition,

$$\mathcal{L}(\mathcal{G}_1 \sqcup \mathcal{G}_2) = \frac{1}{x(\mathcal{G}_1 \sqcup \mathcal{G}_2)} \sum_{P \in \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2)} x(P) y(P)$$

and from the construction of $\text{Graft}_{s,\delta_3}(\mathcal{G}_1, \mathcal{G}_2)$ in Definition 2.5, we see

$$(6.8) \quad x(\mathcal{G}_1 \sqcup \mathcal{G}_2) = \begin{cases} x(\mathcal{G}_3 \sqcup \mathcal{G}_4) \prod_{j=s+1}^{k_4-1} x_{i_j} & \text{if } 0 < s < d; \\ x(\mathcal{G}_3 \sqcup \mathcal{G}_4) & \text{if } s = d; \end{cases}$$

where $k_4 > s + 1$ is the least integer such that $f_1(e_s) = -f_1(e_{k_4-1})$ if such a k_4 exists; and $k_4 = d + 1$ otherwise, and

$$(6.9) \quad x(\mathcal{G}_1 \sqcup \mathcal{G}_2) = \begin{cases} x(\mathcal{G}_5 \sqcup \mathcal{G}_6) \prod_{j=k_5+1}^s x_{i_j} & \text{if } 0 < s < d; \\ x(\mathcal{G}_5 \sqcup \mathcal{G}_6) \left(\prod_{j=k_5+1}^s x_{i_j} \right) \left(\prod_{j=1}^{k_6-1} x_{i'_j} \right) & \text{if } s = d; \end{cases}$$

where $k_5 < s$ is the largest integer such that $f_1(e_{k_5}) = -f_1(e_s)$ if such a k_5 exists; and $k_4 = 0$ otherwise.

Case 1. Suppose first that $0 < s < d$. Using the bijection $\varphi = (\varphi_{34}, \varphi_{56}) : \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2) \rightarrow \text{Match}(\text{Graft}_{s,e_3}(\mathcal{G}_1, \mathcal{G}_2))$ of Theorem 3.1, we obtain

$$\begin{aligned} \mathcal{L}(\mathcal{G}_1 \sqcup \mathcal{G}_2) &= \frac{1}{x(\mathcal{G}_3 \sqcup \mathcal{G}_4) \left(\prod_{j=s+1}^{k_4-1} x_{i_j} \right)} \sum_{\varphi_{34}(P) \in \text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4)} x(P)y(P) \\ &+ \frac{1}{x(\mathcal{G}_5 \sqcup \mathcal{G}_6) \left(\prod_{j=k_5+1}^s x_{i_j} \right)} \sum_{\varphi_{56}(P) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)} x(P)y(P) \end{aligned}$$

Therefore, in case 1, the statement follows from the following lemma. □

Lemma 6.4. (a) *If $\varphi(P) \in \text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4)$ then*

$$x(P)y(P) = x(\varphi_{34}(P))y(\varphi_{34}(P)) \left(\prod_{j=s+1}^{k_4-1} x_{i_j} \right) y_{34}$$

(b) *If $\varphi(P) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$ then*

$$x(P)y(P) = x(\varphi_{56}(P))y(\varphi_{56}(P)) \left(\prod_{j=k_5+1}^s x_{i_j} \right) y_{56}$$

Proof. (a) In the operation $\sigma_{s,3}$ in the definition of φ , we lose an edge of the matching (the grafting edge) with weight $x_{i_{s+1}}$, see Figure 11. The map φ_{34} is given by the first three cases of σ in Figure 11. In each of these cases, the north east vertex of the tile G_s in \mathcal{G}_1 is matched in P by an edge of G_s . The subgraph $\mathcal{G}_1[s, k_4 - 1]$ is a zigzag subgraph of \mathcal{G}_1 , thus, on $\mathcal{G}_1[s + 1, k_4 - 1]$, the matching P is uniquely determined by the fact that the north east vertex of G_s is matched in G_s . Indeed, the matching is as in the left picture of Figure 25. Furthermore, the weight of P

on $\mathcal{G}_1[s + 1, k_4 - 1]$ is equal to $\prod_{j=s+2}^{k_4-1} x_{i_j}$. Therefore

$$x(P)/x(\varphi_{34}(P)) = \left(\prod_{j=s+2}^{k_4-1} x_{i_j} \right) x_{i_{s+1}}$$

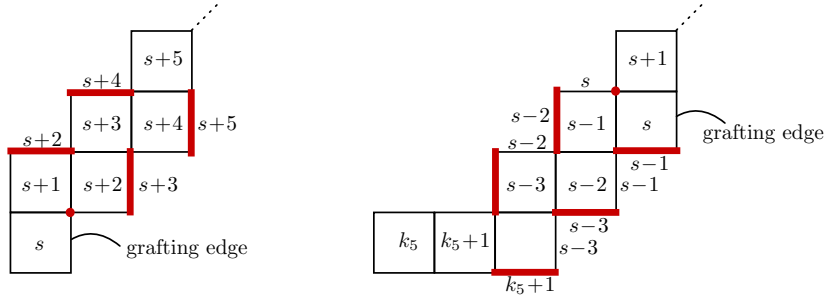


FIGURE 25. Proof of Lemma 6.4: part (a) on the left; part (b) on the right.

where the $x_{i_{s+1}}$ comes from the operation $\sigma_{s,3}$. Moreover, on $\mathcal{G}_1(s+1, k_4-1)$, the matching P is the minimal matching, if the grafting edge is minimal, and P is the maximal matching, otherwise. Thus

$$y(P)/y(\varphi_{34}(P)) = \begin{cases} 1 & \text{if the grafting edge is minimal,} \\ \prod_{j=s+1}^{k_4-1} y_{i_j} & \text{otherwise.} \end{cases}$$

which is exactly the definition of y_{34} . This completes the proof of part (a).

(b) In the operation $\sigma_{s,5}$ in the definition of φ , which is given by the last three cases in Figure 11, we lose an edge (the grafting edge) with weight x_{i_s} . In each of the three cases in the definition of $\sigma_{s,5}$, the northwest vertex of G_s in \mathcal{G}_1 is matched in P by an edge of G_{s+1} . Again, since the subgraph $\mathcal{G}_1[k_5+1, s+1]$ is a zigzag subgraph, then on $\mathcal{G}(k_5+1, s)$ the matching P is uniquely determined. Indeed, the matching is as in the right picture of Figure 25.

Furthermore, the weight of P on $\mathcal{G}(k_5+1, s)$ is equal to $\left(\prod_{j=k_5+1}^{s-1} x_{i_j}\right) x_{i_s}$. Moreover, on $\mathcal{G}(k_5+1, s)$, the matching P is the maximal matching if the grafting edge is minimal, and the minimal matching otherwise. Thus

$$y(P)/y(\varphi_{56}(P)) = \begin{cases} \prod_{j=k_5+1}^{s-1} y_{i_j} & \text{if the grafting edge is minimal} \\ 1 & \text{otherwise} \end{cases}$$

which is exactly the definition of y_{56} . Moreover, comparing this situation to the two cases for $y(P)/y(\varphi_{34}(P))$ above, we see that

$$\begin{aligned} y_{34} = 1 & \iff y_{56} = \prod_{j=k_5+1}^{s-1} y_{i_j} \\ y_{34} = \prod_{j=s+1}^{k_4-1} y_{i_j} & \iff y_{56} = 1 \end{aligned}$$

which completes the proof of part (b).

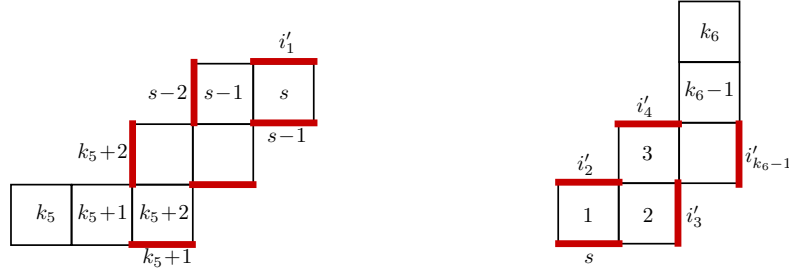


FIGURE 26. Proof of Lemma 6.5

Case 2. Now suppose that $s = d$. Using the bijection $\varphi = (\varphi_{34}, \varphi_{56}) : \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2) \rightarrow \text{Match}(\text{Graft}_{s,e_3}(\mathcal{G}_1, \mathcal{G}_2))$ of Theorem 3.1 and equations (6.8) and (6.9), we obtain

$$\begin{aligned} \mathcal{L}(\mathcal{G}_1 \sqcup \mathcal{G}_2) &= \frac{1}{x(\mathcal{G}_3 \sqcup \mathcal{G}_4)} \sum_{\varphi_{34}(P) \in \text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4)} x(P)y(P) \\ &\quad + \frac{1}{x(\mathcal{G}_5 \sqcup \mathcal{G}_6) \left(\prod_{j=k_5+1}^s x_{i_j} \right) \left(\prod_{j=1}^{k_6-1} x_{i'_j} \right)} \sum_{\varphi_{56}(P) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)} x(P)y(P) \end{aligned}$$

Note that $\mathcal{G}_4 = \{\delta_3\}$ and $x(\mathcal{G}_4) = 1$ in this case. Therefore the statement follows from the following lemma. \square

Lemma 6.5. (a) If $\varphi(P) \in \text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4)$ then

$$x(P)y(P) = x(\varphi_{34}(P))y(\varphi_{34}(P))y_{34}$$

(b) If $\varphi(P) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$ then

$$x(P)y(P) = x(\varphi_{56}(P))y(\varphi_{56}(P)) \left(\prod_{j=k_5+1}^s x_{i_j} \right) \left(\prod_{j=1}^{k_6-1} x_{i'_j} \right) y_{56}.$$

Proof. (a) In the operation $\sigma_{s,3}$ in the definition of φ , we loose an edge of the matching with weight x_{δ_3} where δ_3 is the grafting edge. Since $\mathcal{G}_4 = \{\delta_3\}$ in this case, we get $x(\varphi_{34}(P)) = x(P)$.

Moreover, a direct inspection of the first three cases in Figure 11 (with the tile G_{s+1} removed) shows that

$$y(P) = y(\varphi_{34}(P)),$$

and on the other hand $y_{34} = 1$ since $s = d$.

(b) Since $s = d$, the operation $\sigma_{s,5}$ does not apply and $\varphi_{56}(P)$ is simply given by restricting P to $\mathcal{G}_5 \sqcup \mathcal{G}_6$. Note that $\mathcal{G}_5 = \mathcal{G}_1[1, k_5]$ and $\mathcal{G}_1[k_5 + 1, s]$ is the zigzag graph shown on the left hand side of Figure 26. The matching P is determined on $\mathcal{G}_1[k_5 + 1, s]$ by the initial condition given in the fifth case of Figure 11. Similarly, $\mathcal{G}_6 = \mathcal{G}_2[k_6, d']$ and $\mathcal{G}_2[1, k_6 - 1]$ is shown on the right hand side of Figure 26. Here the matching P is determined by the initial condition as shown in Figure 26. It follows that

$$x(P) = x(\varphi_{56}(P)) \left(\prod_{j=k_5+1}^s x_{i_j} \right) \left(\prod_{j=1}^{k_6-1} x_{i'_j} \right).$$

\square

6.3. Skein relations. As a corollary we obtain a new proof of the skein relations.

Corollary 6.6. *Let γ_1 and γ_2 be two arcs which cross and let (γ_3, γ_4) and (γ_5, γ_6) be the two pairs of arcs obtained by smoothing the crossing.*

(a) *If γ_1 and γ_2 have a non-empty local overlap, then*

$$x_{\gamma_1} x_{\gamma_2} = x_{\gamma_3} x_{\gamma_4} + y(\tilde{\mathcal{G}}) x_{\gamma_5} x_{\gamma_6}$$

where $\tilde{\mathcal{G}}$ is the closure of the overlap \mathcal{G} .

(b) *If γ_1 and γ_2 have an empty local overlap, then*

$$x_{\gamma_1} x_{\gamma_2} = y_{34} x_{\gamma_3} x_{\gamma_4} + y_{56} x_{\gamma_5} x_{\gamma_6}$$

where y_{34} and y_{56} are as in equation 6.7.

Proof. First note that for any $(\ell, k) = (1, 2), (3, 4), (5, 6)$ we have

$$\begin{aligned}
 (6.10) \quad & \mathcal{L}(\mathcal{G}_\ell) \mathcal{L}(\mathcal{G}_k) \\
 &= \frac{1}{x(\mathcal{G}_\ell) y(\mathcal{G}_\ell) x(\mathcal{G}_k) y(\mathcal{G}_k)} \sum_{P_\ell \in \text{Match}(\mathcal{G}_\ell)} x(P_\ell) y(P_\ell) \sum_{P_k \in \text{Match}(\mathcal{G}_k)} x(P_k) y(P_k) \\
 &= \frac{1}{x(\mathcal{G}_\ell) y(\mathcal{G}_\ell) x(\mathcal{G}_k) y(\mathcal{G}_k)} \sum_{(P_\ell, P_k) \in \text{Match}(\mathcal{G}_\ell \times \mathcal{G}_k)} x(P_\ell) x(P_k) y(P_\ell) y(P_k) \\
 &= \frac{1}{x(\mathcal{G}_\ell) y(\mathcal{G}_\ell) x(\mathcal{G}_k) y(\mathcal{G}_k)} \sum_{P \in \text{Match}(\mathcal{G}_\ell \sqcup \mathcal{G}_k)} x(P) y(P) \\
 &= \mathcal{L}(\mathcal{G}_\ell \sqcup \mathcal{G}_k).
 \end{aligned}$$

Therefore in case (a) we have

$$\begin{aligned}
 x_{\gamma_1} x_{\gamma_2} &= \mathcal{L}(\mathcal{G}_1 \sqcup \mathcal{G}_2) \\
 &= \mathcal{L}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)) \\
 &= \mathcal{L}(\mathcal{G}_3 \sqcup \mathcal{G}_4) + y(\tilde{\mathcal{G}}) \mathcal{L}(\mathcal{G}_5 \sqcup \mathcal{G}_6) \\
 &= x_{\gamma_3} x_{\gamma_4} + y(\tilde{\mathcal{G}}) x_{\gamma_5} x_{\gamma_6}
 \end{aligned}$$

where the first equality and the last equality hold by equation (6.10), the second equality by Theorem 6.1, and the third equality by definition.

In case (b) we have

$$\begin{aligned}
 x_{\gamma_1} x_{\gamma_2} &= \mathcal{L}(\mathcal{G}_1 \sqcup \mathcal{G}_2) \\
 &= \mathcal{L}(\text{Graft}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)) \\
 &= y_{34} \mathcal{L}(\mathcal{G}_3 \sqcup \mathcal{G}_4) + y_{56} \mathcal{L}(\mathcal{G}_5 \sqcup \mathcal{G}_6) \\
 &= y_{34} x_{\gamma_3} x_{\gamma_4} + y_{56} x_{\gamma_5} x_{\gamma_6}
 \end{aligned}$$

where the first and the last equality hold by (6.10), the second equality by Theorem 6.3, and the third equality by definition. \square

7. PROOF THAT φ IS A BIJECTION

In this section, we shall construct the inverse map ψ of the map φ of Theorem 3.1

$$\psi : \text{Match}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)) \longrightarrow \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2).$$

We start with the following two lemmas. Recall that the operation ρ is defined in Figure 8.

Lemma 7.1. *If \mathcal{G} is a snake graph and P, P' are two perfect matchings of \mathcal{G} such that the operation ρ does not apply at any of the tiles of \mathcal{G} , then*

$$\begin{aligned}
 P \cup P' &= \{\text{all boundary edges of } \mathcal{G}\}; \\
 P \cap P' &= \emptyset.
 \end{aligned}$$

Proof. If P or P' contains an interior edge then one of the first 3 cases in Figure 8 applies; if P and P' have a common boundary edge, then case 4 applies. \square

Lemma 7.2. *Let \mathcal{G} be a snake graph with sign function f and let P be a perfect matching of \mathcal{G} which consists of boundary edges only. Let NE be the set of all north and east edges of the boundary and SW the set of all south and west edges of the boundary. Then*

$$f(a) = f(b) \quad \text{if } a, b \text{ are both in } P \cap \text{NE or both in } P \cap \text{SW}$$

$$f(c) = -f(b) \quad \text{if one of } a, b \text{ is in } P \cap \text{NE and the other in } P \cap \text{SW}.$$

Proof. We proceed by induction on the number of tiles, d , in \mathcal{G} . If $d = 1$ then the matching consists of either the east and west edge or the north and south edge, and in both cases the result follows directly from the definition of sign functions. Suppose $d > 1$, and let $\mathcal{G}' = \mathcal{G}[1, d-1]$. We may assume without loss of generality that the tile G_d lies east of the tile G_{d-1} . If P contains the east edge of G_d then $P[1, d-1]$ is a perfect matching of \mathcal{G}' , the east edge of G_d has the same sign as the north edge of G_{d-1} and the result follows by induction. Otherwise, P contains the north and the south edge of G_d and $P' = P[1, d-1] \cup \{\text{east edge of } G_{d-1}\}$ is a perfect matching of \mathcal{G}' consisting of boundary edges. By induction, the east edge of G_{d-1} has the same sign as all north and all east edges in P' . The result follows now since the north edge of G_d has the same sign as the east edge of G_{d-1} and the opposite sign of the south of G_d . \square

Now we want to define

$$\psi : \text{Match}(\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)) \longrightarrow \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2).$$

Recall that $\text{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$ is a pair $(\mathcal{G}_3 \sqcup \mathcal{G}_4, \mathcal{G}_5 \sqcup \mathcal{G}_6)$ where \mathcal{G}_3 and \mathcal{G}_4 overlap in \mathcal{G} . Let u, v, u', v' be such that $\mathcal{G} \cong \mathcal{G}_3[u, v] \cong \mathcal{G}_4[u', v']$ is the overlap. Let $P_i \in \text{Match } \mathcal{G}_i$ for $i = 3, 4, 5, 6$. We treat the cases $u \neq v$ and $u = v$ separately.

7.1. Definition of ψ in the case $u \neq v$. Suppose $u \neq 1$ and $u' \neq 1$. Then we define $\psi(P_3, P_4)$ as follows.

- (1) If the pair (P_3, P_4) on $(\mathcal{G}_3[u-1, u+1], \mathcal{G}_4[u'-1, u'+1])$ is one of the eight configurations in Figure 6 then let $\psi(P_3, P_4)$ be

$$(P_3[1, u-1] \cup \mu_{u,1} \cup P_4(u'+1, d_4], \\ P_4[1, u'-1] \cup \mu_{u,2} \cup P_3(u+1, d_3])$$

- (2) If (1) does not apply, let j be the least integer such that $1 < j \leq v-u-2$ and the local configuration of (P_3, P_4) on $(\mathcal{G}_3[u+j, u+j+1], \mathcal{G}_4[u'+j, u'+j+1])$ is one of the four shown in Figure 8, if such a j exists, then let $\psi(P_3, P_4)$ be

$$(P_3[1, u+j-1] \cup \rho_{j,1} \cup P_4(u'+j+2, d_4], \\ P_4[1, u'+j-1] \cup \rho_{j,2} \cup P_3(u+j+2, d_3])$$

- (3) If (1) and (2) do not apply, Lemma 7.3 below implies that μ can be applied to the pair (P_3, P_4) on $(\mathcal{G}_3[v-1, v+1], \mathcal{G}_4[v'-1, v'+1])$, and we let $\psi(P_3, P_4)$ be

$$(P_3[1, v-1] \cup \mu_{v,1} \cup P_4(v'+1, d_4], P_4[1, v'-1] \cup \mu_{v,2} \cup P_3(v+1, d_3])$$

If $u = 1$ or $u' = 1$ then we define $\psi(P_3, P_4)$ using only step (2) with $1 \leq j \leq v-u-1$ and step (3).

Lemma 7.3. *If (1) and (2) do not apply and $u \neq 1$, $u' \neq 1$ then the local configuration of (P_3, P_4) on $(\mathcal{G}_3[v-1, v+1], \mathcal{G}_4[v'-1, v'+1])$ is one of the eight cases on the left in Figure 6, relabeling $s-1, s, s+1$ by $v+1, v, v-1$, respectively, and rotating by 180° .*

Proof. Fix a sign function f on the overlap \mathcal{G} and denote by f_i the induced sign function on \mathcal{G}_i , for $i = 1, 2, 3, 4$. Since \mathcal{G}_1 and \mathcal{G}_2 cross in \mathcal{G} , one of the conditions (i) or (ii) in Definition 2.1 is satisfied. Since $u \neq 1$ and $u' \neq 1$, we must have condition (i), and, because of symmetry, we may assume that $f_1(e_{s-1}) = -f_1(e_t) = \varepsilon$. It follows from the definition of \mathcal{G}_3 and \mathcal{G}_4 that $f_3(e_{u-1}) = f_1(e_{s-1}) = -f_4(e'_{u'-1})$ and $f_3(e_v) = -f_1(e_t) = -f_4(e'_{v'})$. In particular, we have

$$(7.1) \quad f_3(e_{u-1}) = f_3(e_v) = -f_4(e'_{v'}) = \varepsilon$$

Since (1) does not apply, the local configuration of (P_3, P_4) on $(\mathcal{G}_3[u-1, u+1], \mathcal{G}_4[u'-1, u'+1])$ is one of the three cases in Figure 7, relabeling $s-1, s, s+1$ by $u-1, u, u+1$, respectively, where we assume without loss of generality that $\mathcal{G}_3[u-1, u+1]$ is the zigzag graph and $\mathcal{G}_4[u'-1, u'+1]$ is the straight graph. In particular, the north edge of G_u is contained in P_3 , and f_3 equals $-\varepsilon$ on this edge, and the south edge of $G'_{u'+1}$ is contained in P_4 , and f_3 equals $-\varepsilon$ on this edge.

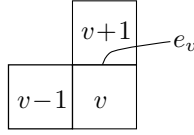
Since (2) does not apply, Lemma 7.1 implies that $P_3[u+1, v-1]$ and $P_4[u'+1, v'-1]$ consist of boundary edges and are disjoint. It then follows from Lemma 7.2 that for all edges $a \in P_3[u+1, v-1]$.

$$f(a) = \begin{cases} \varepsilon & \text{if } a \in P_3 \text{ is a south or a west edge;} \\ -\varepsilon & \text{if } a \in P_3 \text{ is a north or a east edge;} \end{cases}$$

and for all edges $a \in P_4[u'+1, v'-1]$

$$f(a) = \begin{cases} \varepsilon & \text{if } a \in P_4 \text{ is a north or a east edge;} \\ -\varepsilon & \text{if } a \in P_4 \text{ is a south or a west edge.} \end{cases}$$

Now consider the local configuration at G_v . Suppose first that $\mathcal{G}_3[v-1, v+1]$ is a zigzag graph. Without loss of generality assume that G_v lies east of G_{v-1} , as in the following picture.



We know from equation (7.1) that $f_3(e_v) = \varepsilon$. This implies that f_3 equals $-\varepsilon$ on the north edge of G_{v-1} , and therefore the north edge of G_{v-1} is an element of P_3 . Similarly, if the south edge of G_{v-1} is a boundary edge, then it is in P_3 since it has sign ε ; if the south edge of G_{v-1} is not a boundary edge, then G_{v-2} is south of G_{v-1} and the east edge of G_{v-2} has sign $-\varepsilon$, hence is in P_3 . It follows that the southwest corner of G_v is matched by an edge not in G_v . Therefore the east edge of G_v must be contained in P_3 . This implies that the local configuration of (P_3, P_4) on $(\mathcal{G}_3[v-1, v+1], \mathcal{G}_4[v'-1, v'+1])$ is one of the eight configurations in Figure 6 after rotating by 180° and relabeling $v = s$, $v-1 = s+1$, $v+1 = s-1$. This proves the statement in this case.

Now assume that $\mathcal{G}_3[v-1, v+1]$ is straight. Then $\mathcal{G}_4[v'-1, v'+1]$ is zigzag, and we know from equation (7.1) that $f_4(e'_{v'}) = -\varepsilon$. Then an argument similar to the one above shows that the east edge of $G'_{v'}$ belongs to P_4 , and again we conclude that the local configuration is one of the eight in Figure 6. \square

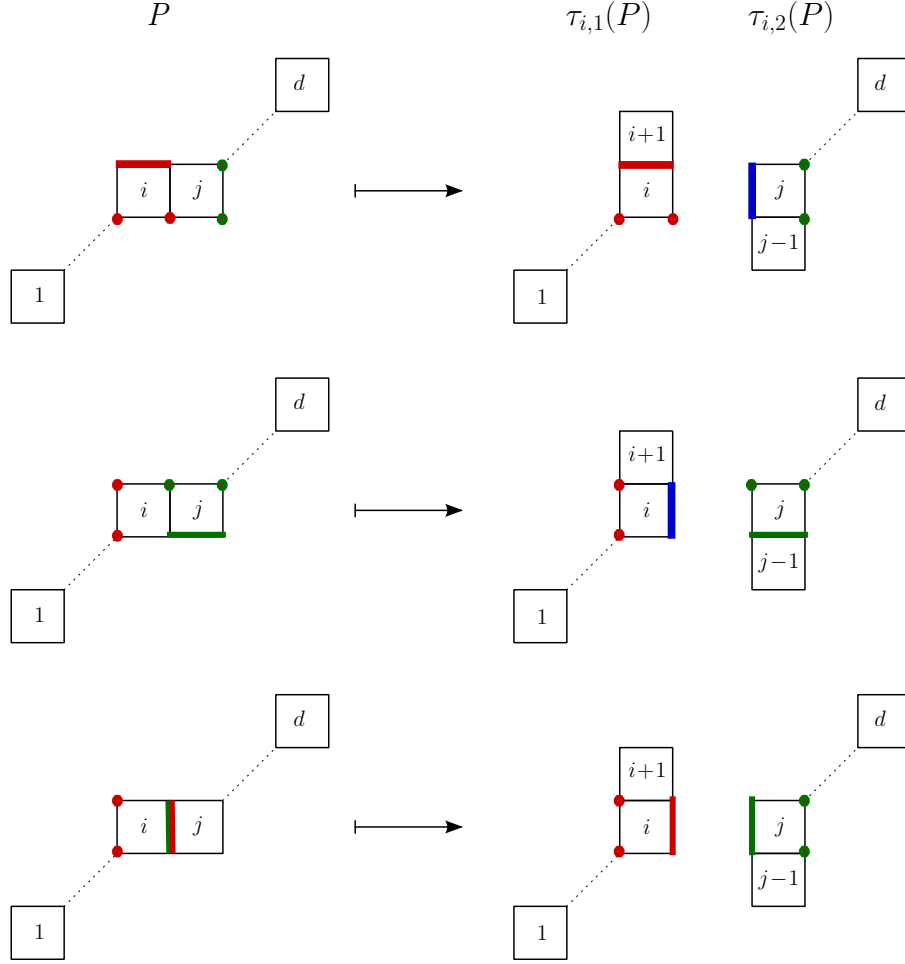
Our next step is to define $\psi(P_5, P_6)$. If $1 < u$, $v' < d_4$, $1 < u'$ and $v < d_3$ then let $\psi(P_5, P_6)$ be

$$(\tau_{u-1,1}(P_5) \cup \eta_1 \cup \tau_{d_3-v,2}(P_6), \tau_{u-1,2}(P_5) \cup \eta_2 \cup \tau_{d_3-v,1}(P_6))$$

where τ is given by Figure 27, and the pair (η_1, η_2) is the unique completion of the matching on $\mathcal{G}_1 \sqcup \mathcal{G}_2$ given by Lemma 7.4 below.

If $u = 1, v' = d_4, 1 = u'$ or $v = d_3$, respectively, then remove the term $\tau_{u-1,1}(P_5)$, $\tau_{d_2-v,2}(P_6)$, $\tau_{u-1,2}(P_5)$ or $\tau_{d_3-v,1}(P_6)$, respectively, from the definition above.

Lemma 7.4. *In the situation above, there exists a unique η_1 and a unique η_2 which consist of boundary edges of \mathcal{G} and which are complementary perfect matchings on the overlap \mathcal{G} .*

FIGURE 27. The operation τ . The new edges are blue.

Proof. The uniqueness of η_1 and η_2 follows from the simple fact that if one chooses an edge a in the first tile of a snake graph then there is a unique way to complete it to a perfect matching consisting only of boundary edges, except possibly for the first edge a . Indeed, for example if $u \neq 1$, the matching of the tile G_{u-1} is given by $\tau_{u-1,1}(P_5)$, and there is one and only one choice to complete a matching on the tile G_u using boundary edges only.

Assume $u \neq 1$ and $u' \neq 1$. There exists a unique way η_1 to extend $\tau_{u-1,1}(P_5)$ into \mathcal{G} using only boundary edges. We need to check that η_1 is compatible with $\tau_{d_3-v,2}(P_6)$. If $d_3 = v$, there is nothing to show, so suppose $v < d_3$. Recall that we assume $u \neq v$. Since \mathcal{G}_3 and \mathcal{G}_4 overlap in \mathcal{G} , we have $f_3(e_{u-1}) = -f_4(e'_{u'-1})$, and since $\mathcal{G}_3, \mathcal{G}_4$ do not cross in \mathcal{G} , we have $f_3(e_{u-1}) = f_3(e_v)$, $f_4(e'_{u'-1}) = f_4(e'_{v'})$, by Definition 2.1. Then, using the definition of $\mathcal{G}_3, \mathcal{G}_4$, we have

$$\begin{aligned}
 f_1(e_{s-1}) &= f_3(e_{u-1}) = f_3(e_v) = f_2(e'_{t'}) = \varepsilon \\
 -f_3(e_{u-1}) &= f_4(e'_{u'-1}) = -\varepsilon \\
 f_1(e_t) &= f_4(e'_{v'}) = f_4(e'_{u'-1}) = -\varepsilon
 \end{aligned}
 \tag{7.2}$$

where $\varepsilon = \pm$. Then we have the following local configuration on \mathcal{G}_1 and \mathcal{G}_2 , respectively.



Since G_{s-1} is matched by $\tau_{u-1,1}(P_5)$, the north edge of G_s is not in P_1 . Using Lemma 7.2, it follows that

$$f_1(a) = \begin{cases} \varepsilon & \text{if } a \in P_1 \cap \mathcal{G} \text{ is a south or a west edge;} \\ -\varepsilon & \text{if } a \in P_1 \cap \mathcal{G} \text{ is a north or a east edge.} \end{cases}$$

In particular, equation (7.2) implies that the east and west edges of G_t are not in η_1 if G_{t+1} is north of G_t in \mathcal{G}_1 , see the left picture in the figure below.



Moreover if G_{t-1} is west of G_t then the north edge of G_{t-1} is not in η_1 , since its sign is ε . Similarly, if G_{t+1} is east of G_t , see the right picture in the figure above, then the north and south edges of G_t are not in η_1 . Moreover, if G_{t-1} is south of G_t then the east edge of G_{t-1} is not in η_1 , since its sign is ε . This shows that the completion η_1 is compatible with $\tau_{d_3-v,2}(P_6)$.

Similarly, since $G'_{s'-1}$ is matched by $\tau_{u-1,2}(P_5)$, the west edge of $G'_{s'}$ is not in P_2 . Therefore

$$f_2(b) = \begin{cases} \varepsilon & \text{if } a \in P_2 \cap \mathcal{G} \text{ is a north or a east edge;} \\ -\varepsilon & \text{if } a \in P_2 \cap \mathcal{G} \text{ is a south or a west edge.} \end{cases}$$

Therefore η_1 and η_2 are complementary on the overlap \mathcal{G} . Again one can show that η_2 is compatible with $\tau_{d_3-v,1}(P_6)$ using the sign conditions. The cases $u = 1$ or $v = 1$ are similar. Note that if $u = 1$ and $v' = d_4$, the matching P_1 is defined by η_1 only and, in this case, the complementarity condition is necessary for the uniqueness of the pair (η_1, η_2) . \square

7.2. Definition of ψ in the case $u = v$. If $u = v$ then $u' = v'$. Let ν^{-1} be the inverse of the map given in the Figure 10. If $u > 1$, $u' > 1$, $d_3 \geq u + 1$, $d_4 \geq u' + 1$ then define $\psi(P_3, P_4)$ as

$$(P_3[1, u-1] \cup \nu_{u,1}^{-1} \cup P_4(u'+1, d_4], P_4[1, u'-1] \cup \nu_{u,2}^{-1} \cup P_3(u+1, d_3])$$

If $u = 1$, $u' = 1$, $d_3 = u + 1$, or $d_4 = u' + 1$, respectively, then remove the term $P_3[1, u-1]$, $P_4[1, u'-1]$, $P_3(u+1, d_3]$, or $P_4(u'+1, d_4]$, respectively, from the definition above, and use the appropriate restrictions of the maps ν^{-1} .

Now we define $\psi(P_5, P_6)$ where $u = v$. If $u > 1$, $u' > 1$, $v < d_3$, $v' < d_4$ then let $\psi(P_5, P_6)$ be

$$(\tau_{u-1,1}(P_5) \cup \tau_{d_3-u,2}(P_6), \tau_{u-1,2}(P_5) \cup \tau_{d_3-u,1}(P_6))$$

Note that both $G_{u-1}, G_u = G'_{u'}, G'_{u'+1}$ and $G'_{u'-1}, G_u = G'_{u'}, G_{u+1}$ form a straight piece. Otherwise, let $\psi(P_5, P_6)$ be

$$\begin{cases} (\eta_1 \cup \tau_{d_3-u,2}(P_6), P_5 \cup \eta_2 \cup \tau_{d_3-u,1}(P_6)) & \text{if } u = 1 \\ (\tau_{u-1,1}(P_5) \cup \tau_{d_3-u,2}(P_6), \tau_{u-1,2}(P_5) \cup \eta_2 \cup P_6) & \text{if } u' = d_4 \\ (P_5 \cup \eta_1 \cup \tau_{d_3-u,2}(P_6), \eta_2 \cup \tau_{d_3-u,1}(P_6)) & \text{if } u' = 1 \\ (\tau_{u-1,1}(P_5) \cup \eta_1 \cup P_6, \tau_{u-1,2}(P_5) \cup \eta_2) & \text{if } u = d_3 \\ (\eta_1 \cup P_6, P_5 \cup \eta_2) & \text{if } u = 1 \text{ and } u' = d_4 \\ (P_5 \cup \eta_1, \eta_2 \cup P_6) & \text{if } u = d_3 \text{ and } u' = 1. \end{cases}$$

where η_i is the unique completion using only boundary edges of \mathcal{G}_i , $i = 1, 2$.

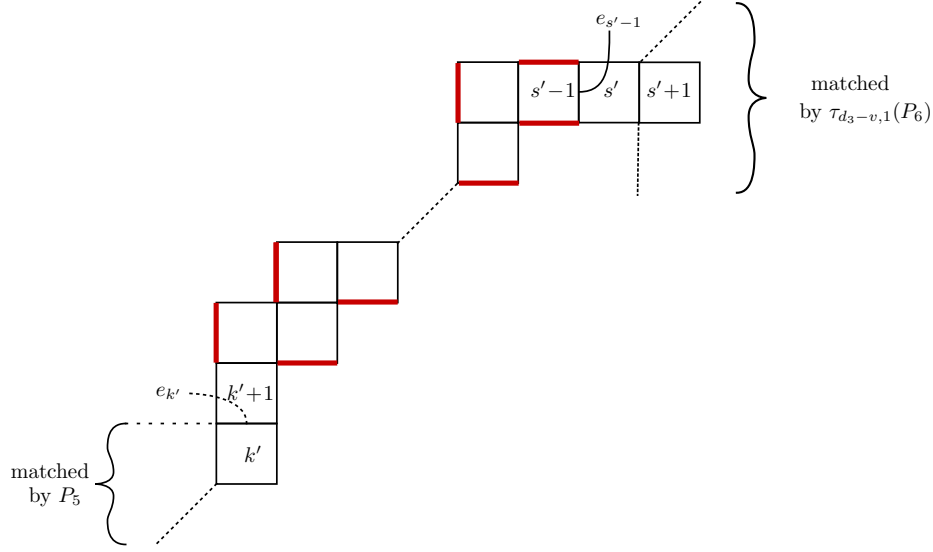
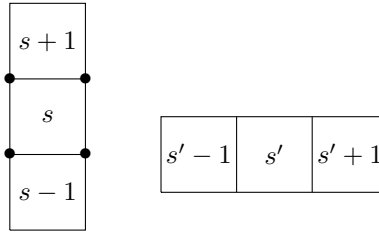


FIGURE 28. Proof of Lemma 7.5

7.3. The map ψ is well-defined. Lemma 7.3 implies that $\psi(P_3, P_4)$ is a perfect matching of $\mathcal{G}_1 \sqcup \mathcal{G}_2$, if $u \neq v$, and, for $u = v$, this follows directly from the definition. Therefore to show that ψ is well-defined it only remains to prove the following Lemma.

Lemma 7.5. $\psi(P_5, P_6)$ is a perfect matching of $\mathcal{G}_1 \sqcup \mathcal{G}_2$.

Proof. If $u \neq v$, this follows from Lemma 7.4. Suppose now that $u = v$. If $u > 1, u' > 1, v < d_3$, and $v' < d_4$, we only need to check that $\psi(P_5, P_6)$ is well-defined on the tiles G_s and $G'_{s'}$ of \mathcal{G}_1 and \mathcal{G}_2 , respectively, which form the overlap. The local configuration of $\mathcal{G}_1[s-1, s+1]$ and $\mathcal{G}_2[s'-1, s'+1]$ is as follows.



The south vertices of G_s are matched by edges of G_{u-1} and the north vertices are matched by the edges of $G_{u'+1}$. Hence, this yields a matching on \mathcal{G}_1 . A similar argument works for \mathcal{G}_2 .

Now suppose $u = 1$. Notice that we still assume $u = v$. Then by definition of ψ in this case, we have

$$\psi(P_5, P_6) = (\eta_1 \cup \tau_{d_3-u,2}(P_6), P_5 \cup \eta_2 \cup \tau_{d_3-v,1}(P_6)).$$

Observe that $\eta_1 \cup \tau_{d_3-u,2}(P_6)$ yields a matching on \mathcal{G}_1 by completion with η_1 . Now we need to show that $P_5 \cup \eta_2 \cup \tau_{d_3-v,1}(P_6)$ is a matching of \mathcal{G}_2 .

Since we have $u = v = 1$, it follows that $s = t = 1$ and $s' = t'$ in Definition 2.3, and therefore $\mathcal{G}_5 = \overline{\mathcal{G}}_2[k', 1]$ where $k' < s' - 1$ is the largest integer such that $f_2(e'_{k'}) = f_2(e'_{s'-1})$. It follows that the subgraph $\mathcal{G}_2[k' + 1, s' - 1]$ is a zigzag subgraph of \mathcal{G}_2 and thus the graph \mathcal{G}_2 has the shape, shown in Figure 28. Note that $\mathcal{G}_2[s' - 1, s' + 1]$ is a straight subgraph.

Moreover $\mathcal{G}_2[1, k']$ is matched by P_5 and the tile $G'_{s'}$ is partially matched by $\tau_{d_3-v,1}(P_6)$. On the remaining zigzag graph $\mathcal{G}_2[k' + 1, s' - 1]$ there is a unique completion η_2 consisting of all west and all south edges on the boundary and the north edge of $G'_{s'-1}$, see Figure 28.

The other cases, $u = d_3$, $u' = 1$, or $u' = d_4$ are similar. \square

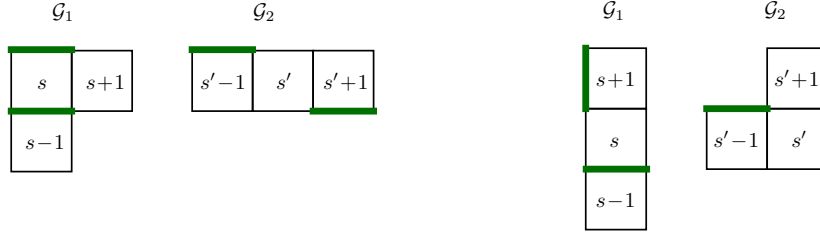


FIGURE 29. Proof of Lemma 7.6

7.4. The map ψ is the inverse of the map φ . It follows immediately from the construction that $\varphi \circ \psi(P_3, P_4) = (P_3, P_4)$ for all $(P_3, P_4) \in \text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4)$ and that $\psi \circ \varphi(P_1, P_2) = (P_1, P_2)$ for all $(P_1, P_2) \in \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2)$ such that $\varphi(P_1, P_2) \in \text{Match}(\mathcal{G}_3 \sqcup \mathcal{G}_4)$.

Now let $(P_1, P_2) \in \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2)$ such that $\varphi(P_1, P_2) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$. This means that $\varphi(P_1, P_2)$ is defined by case (iv) of the definition of φ . In particular, the operation ρ does not apply to the pair (P_1, P_2) and hence Lemma 7.1 implies that (P_1, P_2) consists of boundary edges on \mathcal{G} and are complementary. Let us assume first that $s \neq 1, s' \neq 1, t \neq d$ and $t' \neq d'$, and that $s \neq t$. Then

$$\varphi(P_1, P_2) = (P_1[1, s-1] \sqcup P_2[1, s'-1] \setminus \{a_5\}, P_2[t'+1, d'] \sqcup P_1[t+1, d] \setminus \{a_6\})$$

where a_5 (respectively a_6) is the glueing edge in the definition of \mathcal{G}_5 (respectively \mathcal{G}_6). Since $s \neq t$ we have $u \neq v$ and we must use the definition of ψ given in section 7.1. Note that $\tau_{u-1,1}(P_1[1, s-1] \sqcup P_2[1, s'-1] \setminus \{a_5\})$ is exactly $P_1[1, s-1]$ and that $\tau_{d_3-v,2}(P_2[t'+1, d'] \sqcup P_1[t+1, d] \setminus \{a_6\})$ is exactly $P_1[t+1, d]$, since $d_6 - (d_3 - v) = d - t$. Since the completions η_1, η_2 are unique, it follows from Lemma 7.4 that the first component of $\psi(\varphi(P_1, P_2))$ is equal to P_1 . By a similar argument one can show that the second component of $\psi(\varphi(P_1, P_2))$ is P_2 . If $t = s$ then the overlap consists of a single tile, and the result follows from Lemma 7.5.

In the cases $s = 1, s' = 1, t = d$, or $t' = d'$, a similar argument shows the same result. Note that the edges lost by restricting to $\mathcal{G}_5 \sqcup \mathcal{G}_6$ in the definition of φ are recovered in the completions η_1 and η_2 in the definition of ψ . This shows that $\psi\varphi$ is the identity.

It remains to show that $\varphi\psi(P_5, P_6) = (P_5, P_6)$ for all $(P_5, P_6) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$.

We shall need the following lemma.

Lemma 7.6. *Let $(P_5, P_6) \in \text{Match}(\mathcal{G}_5 \sqcup \mathcal{G}_6)$ and let $(P_1, P_2) = \psi(P_5, P_6)$. Then*

- (a) P_1 and P_2 do not contain interior edges of the overlap \mathcal{G} .
- (b) P_1 and P_2 do not have an edge of \mathcal{G} in common.

Proof. (a) This holds because, by definition, η_i contains only boundary edges of \mathcal{G} .

(b) Using (a) and the fact that \mathcal{G} is a snake graph, it suffices to show that P_1 and P_2 do not have an edge of the first tile of \mathcal{G} in common. Recall that the first tile of \mathcal{G} is denoted by G_s in \mathcal{G}_1 and $G'_{s'}$ in \mathcal{G}_2 and that P_1 is determined on G_s by $\tau_{u-1,2}(P_5)$. We consider the first case of Figure 27. The following two subcases are illustrated in Figure 29. On the left hand side, $\mathcal{G}_1[s-1, s+1]$ is a zigzag graph and $\mathcal{G}_2[s'-1, s'+1]$ is straight. By definition of τ , P_1 contains the south edge of G_s and therefore it also contains the north edge of G_s . Again by definition of τ , P_2 contains the north edge of $G'_{s'-1}$. To see this one needs to flip the rightmost picture in the first row of Figure 27 and relabel j with $s'-1$ and d with 1. Hence P_2 also contains the south edge $G'_{s'+1}$. In particular, this shows (b) in this case.

On the right hand side of Figure 29, $\mathcal{G}_1[s-1, s+1]$ is straight and $\mathcal{G}_2[s'-1, s'+1]$ is zigzag and again P_1 contains the south edge of G_s by definition of τ but now it contains the west edge of G_{s+1} , whereas P_2 contains the north edge of $G'_{s'-1}$ and the east edge of $G'_{s'}$. This shows (b) in this case.

In the remaining two cases of Figure 27, the lemma can be shown by a similar argument. \square

We return to proving that $\varphi\psi(P_5, P_6) = (P_5, P_6)$. Using Lemma 7.6, we show first that $(P_1, P_2) = \psi(P_5, P_6)$ does not satisfy conditions (i),(ii),(iii) of the definition of the map φ . Suppose first that $u > 1$, $u' > 1$, $v' > d_4$ and $v < d_3$, and $u \neq v$.

Lemma 7.6 part (a) excludes cases 4 and 7 of Figure 6 and part (b) excludes cases 2, 3, and 6. In case 5 of the figure, the matching P_1 contains the south edge of G_s and therefore the matching P_5 must be as in the first case of Figure 27. But this implies that P_2 contains the north edge of $G'_{s'-1}$, a contradiction. Finally, in cases 1 and 8 of Figure 6 the matching P_1 contains the west edge of the tile G_s which is not possible by the definition of τ . This shows that (P_1, P_2) does not satisfy condition (i) of the definition of φ .

To show that (P_1, P_2) does not satisfy condition (ii), it suffices to observe that part (a) of Lemma 7.6 excludes the first three cases of Figure 8, and part (b) excludes the fourth case.

To show that (P_1, P_2) does not satisfy condition (iii), we need to exclude all cases of Figure 6 but relabeling $s-1 = t+1$, $s = t$, $s+1 = t-1$, $s'-1 = t'+1$, $s'+1 = t'-1$ and rotating each graph by 180° . Again, part (a) of Lemma 7.6 excludes cases 4 and 7, part (b) excludes cases 2, 3 and 6. In case 5, P_1 contains the common edge of G_t and G_{t+1} , which implies that the matching P_6 must be as in the second case of Figure 27. But then P_2 does not contain the common edge of $G'_{t'}$ and $G'_{t'+1}$, a contradiction. Finally, in cases 1 and 8 of Figure 6, P_1 would contain the east edge of G_t which is not possible by the definition of τ .

We have shown that $(P_1, P_2) = \psi(P_5, P_6)$ does not satisfy any of the conditions (i)-(iii) in the definition of φ , hence condition (iv) applies and we have $\varphi\psi(P_5, P_6) = (P_5, P_6)$.

If $u = v$ then the operation ν does not apply to $\psi(P_5, P_6)$, because, by definition of τ , the vertices on the tiles G_s and $G'_{s'}$ are matched by edges of the tiles $G_{u-1}, G_{u+1}, G'_{u'-1}, G'_{u'+1}$, but each of the pictures on the right hand side of Figure 10 contains at least one edge of G_s or $G'_{s'}$.

In the case $u = 1, u' = 1, v = d_3$ and $v' = d_4$ the result follows by a similar argument. This shows that $\varphi\psi$ is the identity and thus both φ and ψ are bijections; which completes the proof of part (1) of Theorem 3.1.

Next we define the inverse map for part (2)

$$\psi : \text{Match}(\text{Graft}_{s,\delta_3}(\mathcal{G}_1, \mathcal{G}_2)) \longrightarrow \text{Match}(\mathcal{G}_1 \sqcup \mathcal{G}_2)$$

Recall that $\text{Graft}_{s,\delta_3}(\mathcal{G}_1, \mathcal{G}_2)$ is a pair $(\mathcal{G}_3 \sqcup \mathcal{G}_4, \mathcal{G}_5 \sqcup \mathcal{G}_6)$. Let P_3, P_4, P_5, P_6 be perfect matchings of $\mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_6$, respectively.

Let $\psi(P_3, P_4)$ be

$$\begin{aligned} (P_3[1, s-1] \cup (\sigma_{s,3}^{-1})_1 \cup \eta_1 \cup P_4[1, d_4], \eta_2 \cup P_3(s+1, d_3]) & \quad \text{if } s \neq d \\ (P_3[1, s-1] \cup (\sigma_{s,3}^{-1})_1 \cup \eta_1, \eta_2 \cup P_3(s+1, d_3]) & \quad \text{if } s = d \end{aligned}$$

where η_1 is the unique completion using only boundary edges of \mathcal{G}_1 and η_2 is the unique completion using only boundary edges of the first tile of \mathcal{G}_2 .

Let $\psi(P_5, P_6)$ be

$$\begin{aligned} (P_5 \cup \eta_1 \cup (\sigma_{s,5}^{-1})_1 \cup P_6[d'+2, d_6], P_6[1, d'] \cup \eta_2) & \quad \text{if } s \neq d, \\ (P_5 \cup \eta_1, P_6 \cup \eta_2) & \quad \text{if } s = d, \end{aligned}$$

where η_1 is the unique completion using only boundary edges of \mathcal{G}_1 and η_2 the unique completion using only boundary edges of the first tile of \mathcal{G}_2 .

Since the completions in the definition of ψ are unique it follows immediately from the construction that $\varphi\psi$ is the identity and $\psi\varphi$ is the identity. This completes the proof of Theorem 3.1.

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